Answer all 10 questions. Provide all reasoning and show all working. An unjustified answer will receive little or no credit. Begin each question on a separate page. Parts (a) and (b) should be done on the same page if possible. You may write on both sides of the paper.

1. (a) Multiply 15 by 11, & (b) Divide 247 by 19, by using the Egyptian Doubling Method.

2. Find the 3rd rational approximation to the square-root of 2 by using the Babylonian Halving Method and starting with 1 as the 0-th approximation.

3. (a) Find a Pythagorean Triple which contains 11 as its smallest term. (b) Is there a Pythagorean Triple with 4 as its smallest term? (Justify your answer.)

4. Find the greatest common divisor of 171 and 132 by using Euclid’s Algorithm.

5. Let ABC be a Triangle with angle ABC having measure less than 90 degrees. Let AD be the perpendicular to BC from A which meets BC in D. Assuming the Pythagorean Theorem for Right Triangles, prove that $AC^2 = AB^2 + BC^2 - 2 \cdot BC \cdot BD$.

6. Find the first 4 digits of the decimal expansion (3 decimal places) of the square-root of 10 by using the Digit by Digit Algorithm which is essentially due to Theon of Alexandria.

7. (a) How many days were there in the year 1500 CE in Rome, and how many days will there be in the year 2100 CE in Rome? (Explain how you got your answer.) (b) Explain why Pope Gregory found it necessary to let the year 1582 CE have only 355 days.

8. (a) Define what are Axioms, Postulates, Hypothesis, & Theorems in the way that Aristotle understood them. (b) Justify why the curved surface area of a conical frustum is $(\pi) \cdot (R_1 + R_2) \cdot L$ by using the method of infinitesimal slicing.

9. (a) Find the volume of the solid by revolving the disk: $x^2 + y^2 \leq 9$, about the line $x = 5$. (b) Find the centroid of the semicircular curve with equation: $x^2 + y^2 = a^2$ & $x$ non-negative, by using Pappus’ Theorem and the fact that the surface area of a sphere of radius $a$ is $4\pi a^2$. [No integration is necessary to solve this problem.]

10. (a) Define what is the curve known as the Quadratrix of Hippias. (b) The Quadratrix of Hippias can be used to trisect an angle and to square the circle. In what ways are these solutions unsatisfactory?
### MTH 3404 - History of Math

**Solutions to Test #1**

<table>
<thead>
<tr>
<th>1(a)</th>
<th>1</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td>30</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>60</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>120</td>
</tr>
</tbody>
</table>

11 \times 15 = 15 + 30 + 120 = 165

101 | 1 | 19 |
|     | 2 | 38 |
|     | 4 | 76 |
|     | 8 | 152 |

247 - 152 = 95

95 - 76 = 19, 19 - 19 = 0

bec. 11 = 8 + 2 + 1

247 \div 19 = 8 + 4 + 1 = 13

<table>
<thead>
<tr>
<th>2</th>
<th>( A_0 = 1 ), ( A_{k+1} = \left[ \frac{A_k + (n/A_k)}{2} \right] ), ( n = 2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>( A_1 = \frac{1 + 2(1)}{2} = 3/2 )</td>
</tr>
<tr>
<td>2</td>
<td>( A_2 = \left[ \frac{3/2 + 2(3/2)}{2} \right] = (3/2 + 4/3)/2 = 17/12 )</td>
</tr>
<tr>
<td>2</td>
<td>( A_3 = \left[ \frac{(17/12) + 2(17/12)}{2} \right] = (17/12 + 24/17)/2 )</td>
</tr>
<tr>
<td></td>
<td>( = \left[ \frac{(17)(17) + 12(24)}{2(17)(17)} \right] = 577/408 )</td>
</tr>
</tbody>
</table>

3rd rational approx. to \( \sqrt{2} \) is \( 577/408 \)

<table>
<thead>
<tr>
<th>3(a)</th>
<th>( \langle a, b, c \rangle ) is a Pythagorean Triple iff ( \langle a, b, c \rangle ) or ( \langle b, a, c \rangle ) is of the form ( \langle u^2-v^2, 2uv, u^2+v^2 \rangle ) where ( u &amp; v ) are non-negative integers. Take ( u^2-v^2 = 11 ). Then ( u = 6 &amp; v = 5 ). So ( 2uv = 60 ) and ( u^2+v^2 = 61 ). So ( \langle 11, 60, 61 \rangle ) is a Pythagorean triple with 11 as its smallest term.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3(a)</td>
<td>( \langle a, b, c \rangle ) is of the form ( \langle u^2-v^2, 2uv, u^2+v^2 \rangle ) with 4 as its smallest term. Since 4 is the smallest term, ( 2uv &gt; 4 ) ( \Rightarrow u &gt; 0 &amp; v &gt; 0 ). Also since ( u^2-v^2 \geq 4, u &gt; v ). Now if ( 2uv = 4 ), then we must have ( u = 2 &amp; v = 1 ). But this gives ( u^2-v^2 = 2^2 - 1^2 &lt; 4 ). And if ( u^2-v^2 = 4 ), then ( u = 2 &amp; v = 0 ). But this gives ( 2uv &lt; 4 ). So there is no triple with 4 as smallest term.</td>
</tr>
</tbody>
</table>
4. We repeatedly subtract the smaller number until we get a number that is smaller than the smaller number.

\[
\begin{array}{cccccccc}
173 & 132 & 97 & 56 & 39 & 24 & 15 & 9 & 6 & 3 \\
39 & 97 & 56 & 15 & 24 & 9 & 6 & 3 & 0 & 0
\end{array}
\]

: greatest common divisor \((173, 132) = 3 \). Last non-zero number we get.

Another way:

\[
\begin{align*}
173 - 1(132) &= 39 \\
132 - 3(39) &= 15 \\
39 - 2(15) &= 9 \\
15 - 1(9) &= 6 \\
9 - 1(6) &= 3 & \text{gcd} \\
6 - 2(3) &= 0
\end{align*}
\]

5. From the Pythagorean Theorem we get \( AC^2 = AD^2 + DC^2 \)

and \( AB^2 = AD^2 + BD^2 \).

So \( AD^2 = AB^2 - BD^2 \). Also \( DC = BC - BD \).

\[
\begin{align*}
&\text{So } AC^2 = AD^2 + DC^2 = AB^2 - BD^2 + (BC - BD)^2 \\
&= AB^2 - BD^2 + BC^2 - 2BC \cdot BD + BD^2 \\
&= AB^2 + BC^2 - 2BC \cdot BD, \quad \text{quad erat demonstrandum.}
\end{align*}
\]

6. Since \( 1756 < 3162 \), the next digit in the \( \sqrt{10} \) would be less than 5.

So \( \sqrt{10} = 3.162 \) to 3 decimal places.

Note: If \( 1756 \) were > \( 3162 \), then we would have had to increase the last digit we got by 1.
7(a) 1500 CE had 366 days because the Julian Calendar was in use at that time in Rome - so 1500 CE was a leap year. In 2100 CE, the Gregorian Calendar will apply (be in use) - so 2100 CE will have 365 days because it will not be a leap year.

(b) Pope Gregory had to shorten 1582 CE by 10 days because the Julian Calendar was off. According to the Julian calendar, a year is 365 1/4 days - but this is not correct. A year is slightly less than 365 1/4 days - and every 400 years, the Julian Calendar has overestimated the number of days by 3 days. These 3 days every 400 years plus some other discrepancies led Pope Gregory to shorten 1582 CE by 10 days so that the Calendar can be in sync with the seasons.

8(a) An axiom is a self-evident truth - it needs no proof. A postulate is a statement we take to be true so that we can advance in a particular field of study. A hypothesis is a statement that we think might be true, but we have circumstantial evidence supporting it. A theorem is a statement which can be proved from the axioms and postulates.

(b) First cut the curve surface of the frustrum and roll it out into the sector as shown. Then radially divide it in N equal parts. Each part will be like a trapezoid and will have area \( \approx \frac{1}{2} L \left( \frac{2\pi R_1 + 2\pi R_2}{N} \right) = \frac{\pi L (R_1 + R_2)}{N} \).

So the entire area will be \( \approx N \cdot \frac{\pi L (R_1 + R_2)}{N} = \pi L (R_1 + R_2) \).

If these slices are infinitesimally thin (i.e., if N is an infinite non-standard integer) these estimates will be exact.
From Pappus' Theorem 1, we get:

Volume = Area (R) \cdot (Distance travelled by the centroid of R)

= \pi \cdot (3)^2 \cdot 2\pi (5) = 90\pi^2 \text{ cubic units}

(b) From Pappus' Theorem 2, we know:

Area (Sphere) = Length (C) \cdot (Distance travelled by centroid of C)

i.e. \[4\pi a^2 = \frac{1}{2} \cdot 2\pi a \cdot 2\pi x_c\]

So \[x_c = \frac{4\pi a^2}{2\pi a} = \frac{2a}{\pi}\]

Also \[y_c = 0\] by symmetry. \[\therefore\] centroid = \(\left(\frac{2a}{\pi}, 0\right)\).

10(a) Let the line AB fall with uniform velocity so that it reaches OC in 1 second. Also let OA rotate with uniform angular speed so that it reaches OC in 1 second also.

The Quadratrix of Hippasus is the set of single points where the falling line AB intersect the rotating line OA.

(b) Although the Quadratrix is a precisely defined curve, we can only get a finite number of points of it in practice. (This is not the case with the circle because we can draw the whole curve with a pair of compasses.) So if we try to find the intersection of the Quadratrix with a line, we will have to estimate where the intersection is. This makes the solutions of squaring the circle and trisecting an angle unsatisfactory.

Another problem is that we would need a portable drawing of the Quadratrix (say on transparencies) and finally we would need Quadratrix of various sizes to make our constructions of the approp. size.