CHAPTER 1 - LANGUAGES & REG. EXPR.

E1. Let \( A = \{ a \} \) and \( B = \{ b \} \). Describe the following sets:

(a) \( A^* \)
(b) \( A^* \cdot B^* \)
(c) \( (A \cdot B)^* \)
(d) \( (A \cup B)^* \)
(e) \( (A \cap B)^* \)
(f) \( (A \cup B)^* \cdot AB \)
(g) \( (A^* \cup AB)^* \cdot A \cup B^* \)
(h) \( (A^* \cup AB)^* - (B \cap B)^* \)

E2. Find the simplest expressions for the following:

(a) \( \lambda \cdot \phi^* \)
(b) \( \lambda^* \cdot \phi^* \)
(c) \( A^* \cup \phi^* \)
(d) \( (A \cup A)^* \)
(e) \( (\lambda \cup A)^* \)
(f) \( (\lambda^* \cdot \phi^*)^* \)
(g) \( \{ \phi, \{ \phi \} \} - \phi \)
(h) \( \{ \phi, \{ \phi \} \} - \{ \phi \} \)
(i) \( \{ \phi \} \cap \{ \phi, \{ \phi \} \} \)
(j) \( \{ \phi \} \cap \phi \)
(k) \( \lambda - \phi^* \)
(l) \( \lambda - \{ \phi^* \} \)
(m) \( \lambda^* - \phi^* \)
(n) \( \lambda \cdot \phi \)

E3. Find regular expressions for the following languages:

(a) \( \{ \phi \in \{0,1\}^* : 101 \text{ is a substring of } \phi \} \)
(b) \( \{ \phi \in \{0,1\}^* : \text{each } 0 \text{ in } \phi \text{ is immediately followed by at least two } 1's \} \)
(c) \( \{ \phi \in \{0,1\}^* : \text{each } 1 \text{ in } \phi \text{ is immediately followed by the string } 10 \} \)
(d) \( \{ \phi \in \{0,1\}^* : 101 \text{ is not a substring of } \phi \} \)
E4. Say which of the following equations are always true & provide proofs to justify your answers. Also say which of the equations are not always true and provide counterexamples to justify your answers. A, B & C are arbitrary languages.

(a) \( A(BA)^* = (AB)^*A \)
(b) \( (AB)^* = A^*B^* \)
(c) \( A \cdot (B \cdot C)^* = (A \cdot B)^* \cap (A \cdot C)^* \)
(d) \( A^*B = B \cup A^*AB \)
(e) \( (A \cap B)^* = A^* \cap B^* \)
(f) \( A^* \cdot (B \cap C)^* = (AB \cup AC)^* \)
(g) \( (A \cdot B)^* = (A \cup B)^* \)
(h) \( A^* \cdot (B \cup C) = A^* \cdot B \cup A^* \cdot C \)

E5. Let \( S \) be a set of strings of letters from the alphabet \( \mathcal{V} \). We say that \( S \) is commutative if for any \( \alpha \) & \( \beta \) in \( S \)

\[ \alpha \cdot \beta = \beta \cdot \alpha \]

(a) Prove that if \( S \subseteq (\omega)^* \), then \( S \) is commutative. Here \( \omega \) = a fixed string and \( (\omega)^* = \{ \omega^n : n \geq 0 \} \)
(b) Prove that if \( S \) is commutative then we can find a string \( \omega \) such that \( S \subseteq (\omega)^* \)
CHAPTER 4 - REGULAR LANGUAGES

E6. (a) Prove that the language \( \{a^k b^3 k : k \geq 1\} \) is non-regular.
(b) Prove that the language \( \{\psi \psi^R : \psi \in \{0,1\}^*\} \) is non-regular.

E7. A prime number is any positive integer \( p > 1 \) which has only 1 & \( p \) as its divisors. Let \( L = \{a^p : p \text{ is a prime number}\} \). Prove that \( L \) is a non-regular language.

E8. Let \( X \) and \( Y \) be regular languages based on the alphabet \( V \). Determine which of the following languages are always regular and which of them are not always regular.

(a) \( \{w : w \in X \text{ and } w^R \in Y\} \)
(b) \( \{w : w \in X \text{ and } w^R \notin Y\} \)
(c) \( \{w : w \in X \text{ and } w^R = w\} \)
(d) \( \{\psi : \psi \in X \text{ or } \psi \notin Y\} \)
(e) \( \{\psi : \psi \notin X \text{ and } \psi \in X.Y\} \)
E9. Determine which of the following languages are regular and which are non-regular:
(a) \( \{a^k c a^l c a^m : m = k + l \} \)
(b) \( \{a^k c a^l c a^m : m = k + l \pmod{3} \} \)
(c) \( \{a_1 \ldots a_{2n} \in \{0,1\}^* : a_1 \ldots a_n = a_{n+1} \ldots a_{2n} \} \)
(d) \( \{w \in \{0,1\}^* : N_0(w) - N_1(w) \text{ is an even positive integer} \} \)

Here \( N_0(w) = \text{number of 0's in } w \).
and \( N_1(w) = \text{number of 1's in } w \).

E10. (a) Let \( L = \{P^0 Q^n R^n : n \geq 1 \} \) where \( P, Q \& R \)
are non-empty languages based on \( \{0,1\} \).
Is there a choice of \( P, Q \) and \( R \) for which \( L \) is regular?
(b) Find infinite sets \( L_1 \supseteq L_2 \supseteq L_3 \) such that
\( L_1 \& L_3 \) are non-regular and \( L_2 \) is regular.
(c) Find infinite sets \( L_1 \supseteq L_2 \supseteq L_3 \) such that
\( L_1 \& L_3 \) are regular and \( L_2 \) is non-reg.

E11. Suppose \( A \& B \) are non-regular sets. Determine
if it is possible for any of the following
sets to be regular.
(a) \( A - B \) (b) \( A \cup B \) (c) \( A \cdot B \)
CHAPTER 6 - RECURSIVE FUNCTIONS

E12. Define \( \text{sign}(n) = \begin{cases} 0 & \text{if } n = 0 \\
1 & \text{if } n > 1 \end{cases} \) and

\[
\text{sign}(n) = \begin{cases} 1 & \text{if } n = 0 \\
0 & \text{if } n > 1 \end{cases}.
\]

Show that \( \text{sign} \) & \( \text{sign} \) are primitive recursive.

E13. Show that each of the following functions, that are defined below, are primitive recursive.

(i) \( \text{EXP}(x, y) = x^y \)

(ii) \( \text{ABS}(x, y) = |x - y| \)

(iii) \( \text{ZER}(x) = \begin{cases} 1 & \text{if } x = 0 \\
0 & \text{if } x > 0 \end{cases} \)

(iv) \( \text{MIN}(x, y) = \text{SMALLER \ of \ } x \ \text{&} \ y \)

(v) \( \text{MAX}(x, y) = \text{LARGER \ of \ } x \ \text{&} \ y \)

(vi) \( \text{REM}(x, y) = \text{Remainder \ after \ dividing \ } y \text{\ by} \ x \)

(vii) \( \text{QUO}(x, y) = \text{quotient \ obtained \ by \ dividing \ } y \text{\ by} \ x \)

(viii) \( \text{EQU}(x, y) = \begin{cases} 1 & \text{if } x = y \\
0 & \text{if } x \neq y \end{cases} \)

E14. Define \( \text{ls}(x, y) = \begin{cases} 1 & \text{if } x < y \ \text{and} \\
0 & \text{if } x \geq y \end{cases} \)

\[
\text{gr}(x, y) = \begin{cases} 1 & \text{if } x > y \\
0 & \text{if } x \leq y \end{cases}.
\]

Show that \( \text{ls} \) and \( \text{gr} \) are primitive recursive functions.