Answer all 8 questions. An unjustified answer will receive little or no credit. So show all working and provide all justification. NO CALCULATORS ARE ALLOWED.

(15) 1. Let \( f(x) = \frac{x}{3x-1} \) and \( g(x) = \frac{2}{x-1} \).
   a) Find \((f \circ g)(x)\) and \(\text{dom}(f \circ g)\).
   b) Find \((g \circ f)(x)\) and \(\text{dom}(g \circ f)\).

(15) 2. Find a) \( \lim_{x \to 4} \frac{2 - \sqrt{x}}{4 - x} \)
   b) \( \lim_{x \to \infty} \left( \frac{1}{\sqrt{x^2 + 5x - x}} \right) \).

(10) 3. Let \( f(x) = 5x - 13 \) and \( L = 7 \). Prove that \( \lim_{x \to 4} f(x) = L \) by using the \( \varepsilon - \delta \) method.
   [In other words, prove \( \lim_{x \to 4} 5x - 13 = 7 \) the rigorous way.]

(13) 4. a) Find \( \lim_{x \to 3} \sin \left[ \frac{2.\pi(x-3)}{x^2 - 9} \right] \) with adequate justification.
   b) Show that the equation \( x^3 - 3x^2 + 1 = 0 \) has three real roots by using the Intermediate Value Theorem.

PLEASE TURN OVER
5. Find (a) \( \frac{d}{dx} (x \sqrt{1-x^2}) \)  
(b) \( \frac{d}{dx} \left( \frac{\sqrt{2x+3}}{3\sqrt{3x+1}} \right) \) 
and simplify your answers as far as possible.

(12) 6. (a) Let \( f \) be a function. Define what is \( f'(x) \)?  
(b) Let \( f(x) = \frac{1}{x^2} \). Find \( f'(x) \) directly from the definition of \( f'(x) \), (i.e. from first principles).

7. Rapunzel stretches out of the window of her tower and throws a ball vertically upwards. The position of the ball is given by \( y(t) = 25 + 20t - 5t^2 \) when \( t \) is measured in sec. \& \( y \) in metres.  
a) Find the maximum height the ball reaches.  
b) Find the time it takes for the ball to hit the ground. \((y = 0 \text{ at the ground})\).

8. Water is being released from a conical tank at the rate of 36 m\(^3\)/s. If the radius of the tank is 5m \& the height is 10m, find the rate at which the water level is decreasing when the water is 6 m deep.

\[ \text{Spring 2005 'T1, Calc. I} \]
Solutions to Test #1

1(a) \((f \circ g)(x) = f(g(x)) = \frac{g(x)}{3g(x) - 1} = \frac{2/(x-1)}{[3 \cdot 2/(x-1)] - 1}
\[
= \frac{2}{x-1} \cdot \frac{x-1}{7-x} = \frac{2}{7-x}.
\]

Now \(\text{dom}(g) = \mathbb{R} - \{1\}\) and \(\text{dom}(f) = \mathbb{R} - \{4/3\}\).

So \(\text{dom}(f \circ g) = \text{dom}(g) - \text{all } x \text{ such that } g(x) \notin \text{dom}(f)\)
\[
= \mathbb{R} - \{1\} - \text{all } x \text{ such that } \frac{2}{x-1} = \frac{1}{3}
\]
\[
= \mathbb{R} - \{1, 7\} \text{ because }
\]
\[
\frac{2}{x-1} = \frac{1}{3} \Rightarrow 6 = x-1 \Rightarrow x = 7.
\]

(b) \((g \circ f)(x) = g(f(x)) = \frac{2}{f(x) - 1} = \frac{2}{\frac{x}{3x-1} - 1}
\[
= \frac{\frac{2}{x}}{\frac{3x-1}{3x-1}} = \frac{2}{1} \cdot \frac{3x-1}{1-2x} = \frac{2(3x-1)}{1-2x}.
\]

Now \(\text{dom}(g) = \mathbb{R} - \{1\}\) and \(\text{dom}(f) = \mathbb{R} - \{4/3\}\).

So \(\text{dom}(g \circ f) = \text{dom}(f) - \text{all } x \text{ such that } f(x) \notin \text{dom}(g)\)
\[
= \mathbb{R} - \{4/3\} - \text{all } x \text{ such that } \frac{x}{3x-1} = 1
\]
\[
= \mathbb{R} - \{4/3, 1/2\} \text{ because }
\]
\[
\frac{x}{3x-1} = 1 \Rightarrow x = 3x-1 \Rightarrow 1 = 2x \Rightarrow x = 1/2.
\]

2(a) \(\lim_{x \to 4} \frac{2-\sqrt{x}}{4-x} = \lim_{x \to 4} \frac{2-\sqrt{x}}{4-x} \cdot \frac{2+\sqrt{x}}{2+\sqrt{x}}
\]
\[
= \lim_{x \to 4} \frac{4-x}{4-x} \cdot \frac{1}{2+\sqrt{x}} = \lim_{x \to 4} \frac{1}{2+\sqrt{x}}
\]
\[
= \frac{1}{2+\sqrt{4}} = \frac{1}{2+2} = \frac{1}{4}.
\]
2(b) \[
\lim_{x \to +\infty} \frac{1}{\sqrt{x^2 + 5x} - x} = \lim_{x \to +\infty} \frac{1}{\sqrt{x^2 + 5x} + x} \quad \frac{\sqrt{x^2 + 5x} + x}{\sqrt{x^2 + 5x} - x} = \lim_{x \to +\infty} \frac{\sqrt{x^2 + 5x} + x}{X^2 + 5X - X^2} = \lim_{x \to +\infty} \frac{\sqrt{x^2 + 5x} + x}{5X} = \lim_{x \to +\infty} \frac{1}{5} \left( \frac{\sqrt{x^2 + 5x} + x}{X} \right) = \lim_{x \to +\infty} \frac{1}{5} \left( \frac{1 + \frac{5}{X}}{X} + 1 \right) = \frac{1}{5} \left( \frac{1 + 0}{1} + 1 \right) = \frac{2}{5}.
\]

3. Let \(\varepsilon > 0\) be any positive number. Choose \(\delta = \varepsilon/5\). Then for any \(x\) with \(0 < |x - 4| < \delta\) we have
\[
|f(x) - L| = |(5x - 13) - 7| = |5x - 20| = |5(x - 4)| = 5|x - 4| < 5 \cdot \delta = \varepsilon.
\]
Hence for any \(\varepsilon > 0\) we have found a \(\delta > 0\) (namely, \(\delta = \varepsilon/5\)) such that \(|f(x) - 7| < \varepsilon\) whenever \(0 < |x - 4| < \delta\). So \(\lim_{x \to 4} (5x - 13) = 7\).

4(a) \[
\lim_{x \to 3} \sin \left[ \frac{2\pi (x-3)}{x^2 - 9} \right] = \sin \left[ \lim_{x \to 3} \frac{2\pi (x-3)}{x^2 - 9} \right] \quad \text{because \(\sin\) is a continuous function}
\]
\[
= \sin \left[ 2\pi \cdot \lim_{x \to 3} \frac{x-3}{(x-3)(x+3)} \right] = \sin \left[ 2\pi \cdot \lim_{x \to 3} \frac{1}{x+3} \right] = \sin \left( 2\pi \cdot \frac{1}{3 + 3} \right) = \sin \left( \frac{2\pi}{6} \right) = \sin \left( \frac{\pi}{3} \right) = \frac{\sqrt{3}}{2}.
\]
4(b) Let \( f(x) = x^3 - 3x^2 + 1 \). Then

\[
\begin{align*}
  f(-1) &= (-1)^3 - 3(-1)^2 + 1 = -3 \\
  f(0) &= (0)^3 - 3(0)^2 + 1 = 1 \\
  f(1) &= (1)^3 - 3(1)^2 + 1 = -1 \\
  f(3) &= (3)^3 - 3(3)^2 + 1 = 1.
\end{align*}
\]

So it follows by the Intermediate Value Theorem that \( f(x) \) has roots between \(-1\) & 0, between 0 & 1, and between 1 and 3. So \( f(x) \) has at least 3 real roots. Since \( f(x) \) is a polynomial of degree 3, \( f(x) \) can have at most 3 real roots. So \( x^3 - 3x^2 + 1 = 0 \) has exactly 3 real roots.

5(a) \[
\frac{d}{dx} \left( x \cdot \sqrt{1-x^2} \right) = (x)' \cdot (1-x^2)^{1/2} + (x) \cdot \left( (1-x^2)^{1/2} \right)'
\]

\[
= 1 \cdot (1-x^2)^{1/2} + x \cdot \frac{1}{2} \cdot (1-x^2)^{-1/2} \cdot (-2x)
\]

\[
= (1-x^2)^{1/2} - x^2 \cdot (1-x^2)^{-1/2}
\]

\[
= \frac{(1-x^2) - x^2}{(1-x^2)^{1/2}} = \frac{1-2x^2}{\sqrt{1-x^2}}.
\]

5(b) \[
\frac{d}{dx} \left( \frac{\sqrt{2x+3}}{3\sqrt{3x+1}} \right) = \frac{\left( (2x+3)^{1/2} \right)' \cdot (3x+1)^{1/3} - (2x+3)^{1/2} \cdot \left( (3x+1)^{1/3} \right)'}{\left( (3x+1)^{1/3} \right)^2}
\]

\[
= \frac{1}{2} \cdot (2x+3)^{-1/2} \cdot 2 \cdot (3x+1)^{1/3} - (2x+3)^{1/2} \cdot \frac{1}{3} \cdot (3x+1)^{-2/3} \cdot 3
\]

\[
= \frac{1 \cdot (3x+1) - (2x+3) \cdot 1}{(2x+3)^{1/2} \cdot (3x+1)^{2/3} \cdot (3x+1)^{2/3}}
\]

\[
= \frac{3x+1 - 2x - 3}{(2x+3)^{1/2} \cdot (3x+1)^{1/3} \cdot (3x+1)^{1/3}} = \frac{x - 2}{(3x+1)^{1/2} \cdot (3x+1)^{1/3} \cdot (3x+1)^{1/3}}.
\]
6(a) \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \] provided this limit exists. If the limit does not exist, then \( f'(x) \) is undefined.

(b) \[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h} \left[ \frac{1}{(x+h)^2} - \frac{1}{x^2} \right] \]

\[ = \lim_{h \to 0} \frac{1}{h} \frac{x^2 - (x+h)^2}{x^2 (x+h)^2} = \lim_{h \to 0} \frac{1}{h} \frac{x^2 - x^2 - 2xh - h^2}{x^2 (x+h)^2} \]

\[ = \lim_{h \to 0} \frac{1}{h} \frac{h(-2x-h)}{x^2 (x+h)^2} = \lim_{h \to 0} \frac{-2x-h}{x^2 (x+h)^2} \]

\[ = \frac{-2x-0}{x^2 (x+0)^2} = \frac{-2x}{x^2 x^2} = \frac{-2}{x^3} . \]

7(a) When the ball reaches its maximum height, the velocity will be zero. Now \( y(t) = 25 + 20t - 5t^2 \)

\[ \therefore v(t) = y'(t) = 20 - 10t \]

\[ v(t) = 0 \Rightarrow t = 2 . \]

\[ y(2) = 25 + 20(2) - 5(2)^2 = 45 . \]

So the ball will reach a maximum height of 45 m above the ground.

(b) The ball will hit the ground when \( y(t) = 0 \). So

\[ 25 + 20t - 5t^2 = 0 \]

\[ \therefore 5(5 + 4t - t^2) = 0 \]

\[ \therefore 5(5-t)(1+t) = 0 \]

\[ \therefore t = 5 \text{ or } t = -1 . \]

Since the ball was released at time \( t = 0 \), \( t = -1 \) is not possible. So \( t = 5 \). Hence the ball will hit the ground 5 sec. after it is released.
8. We are given that \( \frac{dV}{dt} = -36 \)
We want to find \( \frac{dh}{dt} \mid h=6 \).

We know that \( V = \frac{1}{3} \pi r^2 h \)
\[ \therefore V = \frac{1}{3} \pi \left( \frac{h}{2} \right)^2 h = \frac{\pi h^3}{12} \]
\[ \therefore \frac{dV}{dh} = \frac{\pi h^2}{12} \]
\[ 3h^2 = \frac{\pi h^2}{4} \]

Now \( \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} \) by the Chain Rule. So
\[ -36 = \frac{\pi h^2}{4} \cdot \frac{dh}{dt} \]
\[ \therefore \frac{dh}{dt} = \frac{-36}{\pi h^2/4} = \frac{4 \cdot (-36)}{\pi h^2} \]
\[ \therefore \frac{dh}{dt} \mid h=6 = \frac{4 \cdot (-36)}{\pi \cdot (6)^2} = -\frac{4}{\pi} \text{ m/s} \]

:. the water level is decreasing at the rate of \( \frac{4}{\pi} \) m/s when the water is 6m deep.