Answer all 8 questions. An unjustified answer will receive little or no credit. So show all working and provide all justification. BEGIN EACH QUESTION ON A SEPARATE PAGE. No calculators or formula sheets are allowed.

(12) 1. Find (a) \( \frac{d}{dx} \left( \sqrt{e^{2x} - 1} \right) \) (b) \( \frac{d}{dx} \left( \frac{\tan x}{\sec x - 1} \right) \) and simplify your answers as far as possible.

(12) 2(a) Let \( f(x) = \sqrt{x} \), \( x_0 = 25 \), and \( \Delta x = -0.3 \) using the linear approximation of \( f(x) \) near \( x_0 = 25 \) find \( \sqrt{24.7} \) to two decimal places.

(b) The mass of a spherical planet is given by \( M = \pi r^3/6 \). If the relative error in measuring \( r \) is \( \frac{1}{2} \% \), what is the relative error in calculating \( M \) using this formula.

(10) 3. Suppose \( e^y + y^2 \tan x + \sin(xy) = x^3 \). Find \( dy/dx \) by using implicit differentiation.

(12) 4. (a) Let \( f(x) \) be a function which is differentiable on \( (-\infty, \infty) \). Define what are stationary and inflection points of \( f(x) \).

(b) Find the derivative of \( \sqrt[3]{3x+1}, (x^4 + 2)^{1/4} \sqrt{x^2 + 5} \) by using logarithmic differentiation.
(15) 5. Find (a) \( \lim_{x \to 0} \frac{1 - \cos x}{x \sin x} \) (b) \( \lim_{x \to 0} \left(1 - \frac{x}{2}\right)^{\frac{1}{x}} \)

(14) 6(a) Let \( f(x) \) be a function which is continuous on \((-\infty, \infty)\). Define what is a relative (local) maximum point of \( f \).

(b) Let \( f(x) = 3x^4 + 4x^3 - 12x^2 \). Find all the stationary points of \( f(x) \) and determine which are relative max. pt. & which are rel. min. pt.

(15) 7. Let \( f(x) = x^3 - 3x - 2 \). Make a sketch of the graph of \( f(x) \) and show all the stationary points, inflection points, and \( x \)-intercepts on your graph. Also indicate what are the intervals of increase & decrease, and the intervals of upward and downward concavity.

(16) 8. A conical tank is of height 12 m and has a radius of 4 m. Water is being pumped into the tank at the constant rate of 8 m\(^3\)/s. How fast is the level of the water rising when the depth of the water is 6 m?
1(a) \[ \frac{d}{dx}(\sqrt{e^{2x}-1}) = \frac{d}{dx}\left[(e^{2x})^{1/2}\right] = \frac{1}{2} (e^{2x})^{-1/2} \frac{d}{dx}(e^{2x}) \]
\[ = \frac{1}{2} \cdot \sqrt{e^{2x}} \cdot 2e^{2x} = \frac{e^{2x}}{\sqrt{e^{2x}-1}}. \]

(b) \[ \frac{d}{dx}\left(\frac{\tan x}{\sec x - 1}\right) = \frac{(\tan x)'(\sec x - 1) - \tan x(\sec x - 1)'}{(\sec x - 1)^2} \]
\[ = \frac{(\sec^2 x)(\sec x - 1) - \tan x \cdot \tan x \cdot \sec x}{(\sec x - 1)^2} \]
\[ = \frac{(\sec x - \tan^2 x) \cdot \sec x - \sec^2 x}{(\sec x - 1)^2} \]
\[ = \frac{\sec x}{(\sec x - 1)} = \sec x. \]

2(a) \[ \sqrt{24.7} = \sqrt{x_0 + \Delta x} = f(x_0 + \Delta x) \]
\[ f(x) = \sqrt{x} \]
\[ = f(x_0) + \Delta x \cdot f'(x_0) \]
\[ = \sqrt{x_0} + (0.3)/2\sqrt{x_0} \]
\[ = 5.00 - 0.3/10 = 4.97 \]

(b) \[ M = \pi r^5/6. \quad \text{So} \quad \frac{dM}{dr} = 5\pi r^4/6. \quad \frac{dM}{M} = \frac{5\pi r^4 dr}{6} \]
\[ \therefore \frac{dM}{M} = \frac{5\pi r^4 dr}{6} \]
\[ \frac{6}{\pi r^5/6} = \frac{6}{\pi r^5/6} = \frac{5\pi r^4 dr}{6} \]
\[ = \frac{5}{\pi r^5/6} \]

Since \[ \frac{dr}{r} = \frac{1}{2}, \quad \frac{dM}{M} = 5\left(\frac{1}{2}\right) = 21/2 %. \]
So relative error in the calculation of \( M \) is 21/2 %

3. \[ \frac{d}{dx}\left[e^y + y^2 \cdot \tan x + \sin(xy)\right] = \frac{d}{dx} (x^3) \]
\[ \therefore e^y \frac{dy}{dx} + (y^2)' \cdot \tan x + y^2 (\tan x)' + \cos(xy)(xy)' = 3x^2 \]
\[ \therefore e^y \frac{dy}{dx} + 2y \cdot \tan x \cdot \frac{dy}{dx} + y^2 \cdot \sec^2 x + \cos(xy) \left[1 + y \cdot \frac{dy}{dx}\right] = 3x^2 \]
3. \[ e^y \frac{dy}{dx} + 2y \tan x \frac{dy}{dx} + x \cos(xy) \frac{dy}{dx} = 3x^2 - y^2 \sec^2 x - y \cos(xy) \]
\[ \therefore \frac{dy}{dx} = \frac{3x^2 - (y \sec x)^2 - y \cos(xy)}{e^y + 2y \tan x + x \cos(xy)} \]

4(a) A stationary point of \( f(x) \) is any point \( x_0 \) with \( f'(x_0) = 0 \). An inflection point of \( f(x) \) is any point \( x_0 \) at which the graph of \( y = f(x) \) changes its concavity.

(b) Let \( y = (3x+1)^{1/3} (x^4 + 2)^{1/4} / \sqrt{x^2 + 5} \). Then
\[ \ln(y) = \ln((3x+1)^{1/3}) + \ln((x^4 + 2)^{1/4}) - \ln((x^2 + 5)^{1/2}) \]
\[ \therefore \ln(y) = \frac{1}{3} \ln(3x+1) + \frac{1}{4} \ln(x^4 + 2) - \frac{1}{2} \ln(x^2 + 5) \]
\[ \therefore \frac{d}{dx} \ln(y) = \frac{1}{3} \cdot \frac{3}{3x+1} + \frac{1}{4} \cdot \frac{4x^3}{x^4 + 2} - \frac{1}{2} \cdot \frac{2x}{x^2 + 5} \]
\[ \therefore \frac{1}{y} \frac{dy}{dx} = \frac{1}{3x+1} + \frac{x^3}{x^4 + 2} - \frac{x}{x^2 + 5} \]
\[ \therefore \frac{dy}{dx} = \frac{(3x+1)^{1/3} (x^4 + 2)^{1/4}}{\sqrt{x^2 + 5}} \left[ \frac{1}{3x+1} + \frac{x^3}{x^4 + 2} - \frac{x}{x^2 + 5} \right] \]

5(a) \[ \lim_{x \to 0} \frac{1 - \cos x}{x \sin x} = \lim_{x \to 0} \frac{0 - (-\sin x)}{1 \cdot \sin x + x \cdot \cos x} \]
\[ = \lim_{x \to 0} \frac{\sin x}{\sin x + x \cos x} = \lim_{x \to 0} \frac{\cos x}{\cos x + x \cdot \cos x + x \cdot (-\sin x)} \]
\[ = \frac{\cos(0)}{\cos(0) + \cos(0) + 0(-0)} = \frac{1}{1+1} = \frac{1}{2} \]

(b) Let \( L = \lim_{x \to 0} (1 - \frac{x}{2})^{\frac{1}{x}} \). Then \[ \ln(L) = \ln\left[ \lim_{x \to 0} (1 - \frac{x}{2})^{\frac{1}{x}} \right] \]
\[ = \lim_{x \to 0} \ln\left(1 - \frac{x}{2}\right)^{\frac{1}{x}} \] because \( \ln \) is a continuous function.
\[ = \lim_{x \to 0} \frac{\ln(1 - \frac{x}{2})}{x} = \lim_{x \to 0} \frac{-\frac{1}{12}}{x \frac{1 - \frac{x}{2}}{1}} \] (by the \( 0 \) \( 0 \) rule) = \( \frac{1}{2} \).

Hence \( \ln(L) = -\frac{1}{2} \) so \( L = e^{-\frac{1}{2}} = \frac{1}{\sqrt{e}} \).

6(a) A point \( x_0 \) is a relative maximum point of \( f \) if we can find an open interval \( I \) containing \( x_0 \) such that \( f(x) \leq f(x_0) \) for all \( x \) in \( I \).
6(b) \[ f(x) = 3x^4 + 4x^3 - 12x^2 \]
\[ f'(x) = 12x^3 + 12x^2 - 24x = 12x(x^2 + x - 2) = 12x(x - 1)(x + 2) \]
So the stationary points are \( x_0 = -2 \), \( x_1 = 0 \) & \( x_2 = 1 \).
\[ f''(x) = 36x^2 + 24x - 24 = 12(3x^2 + 2x - 2) \]
\[ f''(-2) = 12[3(-2)] + 2(-2) - 2) = 72 > 0 \quad \text{So } x_0 = -2 \text{ is rel. min pt.} \]
\[ f''(0) = 12[3(0) + 2(0) - 2] = -24 < 0 \quad \text{So } x_1 = 0 \text{ is rel. max pt.} \]
\[ f''(1) = 12[3(1) + 2 - 2] = 36 > 0 \quad \text{So } x_2 = 1 \text{ is rel. min pt.} \]

7 \[ f(x) = x^3 - 3x - 2 \]
\[ f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1) \]
\[ f''(x) = 6x \]
\[ f(-1) = (-1)^3 - 3(-1) - 2 = 0 \]
\[ \therefore x - (-1) = x + 1 \text{ is a factor of } f(x) \]
\[ x^3 - 3x - 2 = (x + 1)(x^2 - x - 2) = (x + 1)(x + 1)(x - 2) \]

Stationary points are \(-1\) and 1
Inflection point is \( x_0 = 0 \)
\( x \)-intercepts are at \(-1\) and 2.

8. We are given that \( \frac{dV}{dt} = \frac{m^3}{s} \).
We want to find \( \frac{dh}{dt} \) at \( h = 6 \).
Now \( V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left(\frac{1}{2}h^2 \cdot h = \frac{\pi h^3}{27} \right. \]
\[ \therefore \frac{dV}{dh} = \frac{3\pi h^2}{27} = \frac{\pi h^2}{9} \]
But \( \frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt} \) (Chain Rule)
So \( 8 = \frac{\pi h^2}{9} \cdot \frac{dh}{dt} \)
\[ \therefore \frac{dh}{dt} \Bigg|_{h=6} = \frac{9}{\pi h^2} \Bigg|_{h=6} = \frac{1}{\pi \cdot 36} = \frac{8}{21} \text{ m/s} \]