Answer all 8 questions. An unjustified answer will receive little or no credit. No calculators or formula sheets are allowed. BEGIN EACH QUESTION ON A SEPARATE PAGE.

[12] (a) Evaluate \( \int_0^2 \frac{3x^3}{4} \, dx \) by finding the Riemann sum \( \sum_{k=1}^{n} f(x_k^*) \Delta x \) and letting \( n \to \infty \). Here \((x_0, x_1, \ldots, x_n)\) is the uniform partition of \([0, 2]\) and \(x_k^*\) is the right endpoint of each sub-interval.
(b) Check your answer by using anti-derivatives.

[13] (a) Write down what the two parts of the Fundamental Theorem of Calculus say.
(b) Find \( \frac{d}{dx} \left[ \int_x^{\sin x} \sqrt{1+e^{2t}} \, dt \right] \). Justify your answer completely.

[12] (a) A particle moves along the \( x \)-axis with velocity given by \( v(t) = (4t - t^3) \) \( \text{m/s} \).
(b) Find the displacement from time \( t = 0 \) to time \( t = 4 \) sec.
(b) Find the total distance travelled between \( t = 0 \) and \( t = 4 \) sec.

[15] Let \( R \) be the region bounded by the curves \( y = 5x^2 + 3 \), \( y = 0 \), \( x = 2 \), and \( x = 0 \). Make a sketch and then find the volume of the solid formed by
(a) revolving \( R \) about the \( x \)-axis.
(b) revolving \( R \) about the \( y \)-axis.
5(a) Define what is an anti-derivative of f over \([a,b]\).
(b) Find the anti-derivative \( \int x^3 \sqrt{1-x^2} \, dx \) by using an appropriate substitution.

6(a) Find the total area enclosed by the curves \( y = x^2 \) and \( y = x + 2 \).
(b) Find the length of the curve \( y = \frac{1}{x} + \frac{x^3}{12} \) as \( x \) varies from 1 to 2.

7. Let \( C \) be the curve \( y = \frac{x^2}{8} \) as \( x \) varies from 0 to 3. Make a sketch and then find the area of the curved surface formed by revolving \( C \) about the \( y \)-axis.

8. A conical tank of radius 3m and height 6m is filled with water to a depth of 2m. How much work must be done to pump all of the water over the rim of the tank?

\[ \text{Use density of water} = 10^3 \text{ kg/m}^3 \text{ and acceleration due to gravity, } g = 10 \text{ m/s}^{-2} \]

FA’09 Test #1
1(a) \[ \Delta x = (b-a)/n = (2-0)/n = 2/n, \quad [a, b] = [0, 2], \]
\[ x_k = a + k \cdot \Delta x = 0 + 2k/n = 2k/n, \quad f(x) = \frac{3}{4} x^3. \]
\[ x^*_k = \text{right endpoint} = x_k = 2k/n. \quad \therefore \]
\[ \sum_{k=1}^{n} f(x^*_k) \cdot \Delta x_k = \sum_{k=1}^{n} \frac{3}{4} \left( \frac{2k}{n} \right)^3 \cdot \frac{2}{n} = \frac{3}{4} \cdot \frac{2}{n} \cdot \left( \frac{2}{n} \right)^3 \sum_{k=1}^{n} k^3 \]
\[ = \frac{3 \cdot 2 \cdot 8}{4 \cdot n^4} \left[ \frac{n(n+1)}{2} \right]^2 = \frac{12}{2^2} \cdot \frac{n \cdot n \cdot (n+1)(n+1)}{n \cdot n \cdot n \cdot n} \]
\[ = 3 \left( 1 + \frac{1}{n} \right) \left( 1 + \frac{1}{n} \right) \to 3 \quad \text{as} \quad n \to \infty \]
\[ \therefore \int_0^2 \frac{3}{4} x^3 \, dx = 3. \]

(b) \[ \int_0^2 \frac{3}{4} x^3 \, dx = \left[ \int \frac{3}{4} x^3 \, dx \right]_0^2 = \left[ \frac{3}{4} \cdot \frac{x^4}{4} \right]_0^2 = \frac{3}{4} \cdot \frac{16}{4} = 3. \]

2(a) (i) If \( f \) is a continuous function on \([a, b]\) and \( F \) is any anti-derivative of \( f \) over \([a, b]\), then \( \int_a^b f(x) \, dx = F(b) - F(a) \)
(ii) If \( f \) is a continuous function on \( I \) and \( a \) is any fixed point in \( I \), then \( \frac{d}{dx} \left[ \int_a^x f(t) \, dt \right] = f(x) \) for each \( x \) in \( I \).

(b) \[ \frac{d}{dx} \left[ \int_x^{\sin x} \sqrt{1+e^{2t}} \, dt \right] = \frac{d}{dx} \left[ \int_x^0 \sqrt{1+e^{2t}} \, dt + \int_0^{\sin x} \sqrt{1+e^{2t}} \, dt \right] \]
\[ \text{Put } u = \sin x \]
\[ \frac{du}{dx} = \cos x \]
\[ \frac{d}{dx} \left[ \int_0^{\sin x} \sqrt{1+e^{2t}} \, dt \right] - \frac{d}{dx} \left[ \int_0^x \sqrt{1+e^{2t}} \, dt \right] \]
\[ = \frac{d}{dx} \left[ \int_0^{u} \sqrt{1+e^{2t}} \, dt \right] - \sqrt{1+e^{2u}} \]
\[ = \frac{d}{du} \left[ \int_0^{u} \sqrt{1+e^{2t}} \, dt \right] \cdot \frac{du}{dx} - \sqrt{1+e^{2u}} \]
\[ = \sqrt{1+e^{2u}} \cdot \frac{du}{dx} - \sqrt{1+e^{2x}} = \cos x \cdot \sqrt{1+e^{2\sin x}} - \sqrt{1+e^{2x}} \]

3(a) Displacement = \[ \int_0^4 v(t) \, dt = \int_0^4 (4t - t^3) \, dt \]
\[ = \left[ \frac{4t^2}{2} - \frac{t^4}{4} \right]_0^4 = 2(4)^2 - 4^3 = 32 - 64 = -32 \text{ m}. \]
3 (b) \( v(t) = 0 \Rightarrow t(4-t^2) = 0 \Rightarrow t(2-t)(2+t) = 0 \Rightarrow t = 0, -2, 2 \)

So \( |v(t)| = \begin{cases} 
4t - t^3 & 0 \leq t \leq 2 \\
2t^3 - 4t & 2 < t \leq 4 .
\end{cases} \)

Total distance travelled = \( \int_0^4 |v(t)| \, dt \)
= \( \int_0^2 (4t - t^3) \, dt + \int_2^4 (2t^3 - 4t) \, dt \)
= \( \left[ 2t^2 - \frac{t^4}{4} \right]_0^2 + \left[ \frac{t^4}{4} - 2t^2 \right]_2^4 \)
= \( \left[ 2(2)^2 - \frac{2^4}{4} \right] - [0] + \left( \frac{4^4}{4} - 2(4)^2 \right) - \left( \frac{2^4}{4} - 2(2)^2 \right) \)
= \( (8 - 4) + (64 - 32) - (4 - 8) = 40 \) m.

4 (a) \( V_1 = \int_0^2 \pi y^2 \, dx \)
= \( \int_0^2 \pi \left( 5x^2 + 3 \right)^2 \, dx \)
= \( \pi \int_0^2 (25x^4 + 30x^2 + 9) \, dx \)
= \( \pi \left[ \frac{25}{5} x^5 + \frac{30}{3} x^3 + 9x \right]_0^2 \)
= \( \pi \left[ 5(2)^5 + 10(2)^3 + 9(2) \right] - [0] = 258\pi \)

(b) \( V_2 = \int_0^2 2\pi x \cdot y \, dx \)
= \( \int_0^2 2\pi x \left( 5x^2 + 3 \right) \, dx \)
= \( 2\pi \int_0^2 (5x^3 + 3x) \, dx \)
= \( 2\pi \left[ \frac{5}{4} x^4 + \frac{3}{2} x^2 \right]_0^2 \)
= \( 2\pi \left[ 5(2)^4 + 3(2)^2 \right] = 2\pi(52) = 52\pi . \)

5 (a) An anti-derivative of \( f \) over \([a,b]\) is any function \( F \) on \([a,b]\) such that \( F'(x) = f(x) \) for each \( x \) in \([a,b]\).

(b) \( \int x^3 \sqrt{1-x^2} \, dx = \int x^2 \cdot \frac{1}{2} \cdot \sqrt{1-x^2} \cdot 2x \, dx \)

Put \( u = 1-x^2 \)
\( du = -2x \, dx \)
\( x^2 = 1-u \)

\( \int x^3 \sqrt{1-x^2} \, dx = - \frac{1}{2} \int (1-u) \sqrt{u} \cdot du = \frac{1}{2} \int (u^{3/2} - u^{5/2}) \, du \)

\( = \frac{1}{2} \left[ \frac{2}{5} u^{5/2} - \frac{2}{3} u^{3/2} \right] + C = (1-x^2)^{5/2} - (1-x^2)^{3/2} + C. \)
6. \[ \frac{d}{dx} (x^2 + xz) = 2x + z \]

\[ \frac{d}{dx} (x^2 + xz) = \frac{d}{dx} (x^2) + \frac{d}{dx} (xz) \]

\[ = 2x + z \]

\[ \int_{-2}^{2} (y_2 - y_1) \, dx = \int_{-2}^{2} (x + 2x - x^2) \, dx \]

\[ = \left[ \frac{x^2}{2} + 2x - \frac{x^2}{3} \right]_{-2}^{2} = \left[ \frac{4}{2} + 2(2) - \frac{8}{3} \right] - \left[ \frac{4}{2} - 2(-2) + \frac{8}{3} \right] = \frac{9}{2} \]

(b) \[ y = \left( \frac{1}{x} \right) + x^{3/2} \]

\[ \frac{dy}{dx} = -\frac{1}{x^2} + \frac{3x^{1/2}}{4} \]

\[ 1 + \left( \frac{dy}{dx} \right)^2 = 1 + \left( -\frac{1}{x^2} + \frac{x^{1/2}}{4} \right)^2 \]

\[ = 1 + \left( \frac{1}{x^2} \right)^2 - \frac{1}{2} + \left( \frac{x^{1/2}}{4} \right)^2 \]

\[ = \left( \frac{1}{x^2} \right)^2 + \frac{1}{2} + \left( \frac{x^{1/2}}{4} \right)^2 = \left( \frac{1}{x^2} + \frac{x^{1/2}}{4} \right)^2 \]

\[ L = \int_{1}^{2} \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = \int_{1}^{2} \left( \frac{1}{x^2} + \frac{x^{1/2}}{4} \right) \, dx \]

\[ = \left[ -\frac{1}{x} + \frac{x^{3/2}}{12} \right]_{1}^{2} = \left( -\frac{1}{2} + \frac{8}{12} \right) - \left( -1 + \frac{1}{12} \right) = \frac{13}{12} \]

7. \[ S = \int_{0}^{3} 2\pi x \cdot dl = 2\pi \int_{0}^{3} x \cdot \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \cdot dx \]

\[ = 2\pi \int_{0}^{3} x \cdot \sqrt{1 + \frac{x^2}{16}} \cdot dx \]

\[ = 2\pi \int_{1}^{2} \left( u^{1/2} \right) \cdot 8 \, du \]

\[ = 16\pi \cdot \frac{2}{3} \left[ u^{3/2} \right]_{1}^{2} \]

\[ = 32\pi \cdot \frac{2}{3} \left( \frac{25}{16} \cdot \frac{5}{4} - 1 \right) \]

\[ = \frac{32\pi}{3} \cdot \left( \frac{25}{64} - 1 \right) = \frac{32\pi}{3} \cdot \frac{61}{64} = \frac{61\pi}{6} \]

8. \[ W = \int_{x=0}^{2} \text{d}W = \int_{x=0}^{2} \text{Force} \cdot \text{dist} \]

\[ = \int_{x=0}^{2} (dM) \cdot g \cdot (6-x) \]

\[ = \int_{0}^{2} 10^3 \pi r^2 \, dx \cdot 10 \cdot (6-x) \]

\[ = 10^4 (\pi/4) \cdot \int_{0}^{2} (6x^2 - x^3) \, dx \]

\[ = 10^4 (\pi/4) \cdot \left[ 6 \cdot \frac{x^3}{3} - \frac{x^4}{4} \right]_{0}^{2} \]

\[ = 10^4 (\pi/4) \cdot (16 - 4) = 3\pi \cdot 10^4 \text{Joules} \]