1. (a) Define ordinal exponentiation by transfinite recursion.
   (b) Prove that for any ordinals \( \alpha, \beta, \gamma \),
   \( \alpha^{\beta + \gamma} = \alpha^\beta \cdot \alpha^\gamma \).
   (You may use any results needed about ordinal add. & mult.)

2. Simplify the following ordinal expressions as far as possible
   (a) \( 3 + (\omega_2 + \omega^2) \)
   (b) \( (\omega + 3)(\omega + 2) \)

3. (a) Define what are the following cardinal operations
        \( K + M \), \( K \cdot M \) and \( K^M \).
   (b) Prove that \( K + K = K \) if \( K \) is an infinite cardinal.

4. (a) Write down what the Cantor-Bernstein Theorem says and define what \( A \leq B \) means.
   (b) Let \( \text{Seq}(\mathbb{N}) \) = the set of all finite sequences of elements of \( \mathbb{N} \). Prove that \( \text{Seq}_f(\mathbb{N}) \approx \mathbb{N} \).

5. (a) Write down what the Axiom of Choice (AC) says & what is the Well-ordering Principle (W.O.P).
   (b) Prove that W.O.P. \( \Rightarrow \) AC.

6. (a) Let \( \lambda \) be a limit ordinal. Define what is \( \text{col}(\lambda) \).
       Also define what is a strongly inaccessible cardinal.
   (b) Prove that if \( \lambda \) is a limit ordinal then \( \text{col}(\lambda) \)
       is always a cardinal.
1 (a) We define $x^\beta$ by transfinite recursion as follows

(i) $x^0 = 1$
(ii) $x^{\beta+1} = x^\beta \cdot x$, and
(iii) $x^\lambda = \sup \{x^\beta : \beta < \lambda \}$ for limit ordinals $\lambda > 0$.

(b) We will prove $x^{\beta+\gamma} = x^\beta \cdot x^\gamma$ by transfinite induction on $\gamma$. Here $x$ and $\beta$ are fixed but arbitrary. For $\gamma = 0$ we have, $x^{\beta+0} = x^\beta \cdot x_0^0 = x^\beta \cdot 1 = x^\beta$. So the result is true for $\gamma = 0$.

Now suppose the result is true for $\gamma$. We will prove it for $\gamma+1$. We have

$x^{\beta+1} = x^{(\beta+\gamma)+1}$

bec. add. is assoc.

$= x^{\beta+\gamma} \cdot x$ by (ii) above

$= (x^\beta \cdot x^\gamma) \cdot x$ bec. result is true for $\gamma$

$= x^\beta \cdot (x^\gamma \cdot x)$ bec. mult. is assoc.

$= x^\beta \cdot x^{\gamma+1}$ by (ii) above again.

So if the result is true for $\gamma$, then it will be true for $\gamma+1$.

Finally suppose the result is true for all $\gamma < \lambda$ where $\lambda$ is a limit ordinal $> 0$. Then

$x^{\beta+\lambda} = \sup \{x^{\beta+\gamma} : \gamma < \lambda \}$ bec. $\beta+\lambda$ is lim. ord.

$= \sup \{x^\beta \cdot x^\gamma : \gamma < \lambda \}$ bec. result is true for $\gamma < \lambda$

$= x^\beta \cdot \sup \{x^\gamma : \gamma < \lambda \}$ bec. sup of sup $\{x^\gamma : \gamma < \lambda \}$ is lim. ord.

$= x^\beta \cdot x^\lambda$ bec. $x^\lambda = \sup \{x^\gamma : \gamma < \lambda \}$

So if the result is true for all $\gamma < \lambda$, then it will be true for $\lambda$. Hence by the principle of Transfinite Induction, it is true for all $\gamma$. Since $x$ and $\beta$ were arbitrary, it will be true for all $x, \beta$ and $\gamma$. 
2(a)  
\[ 3 + w \cdot 2 + w^2 = ((3 + w) \cdot w) + w^2 = (\sup\{3 + n : n < w\} + w) + w^2 = (w + w) + w^2 = \omega \cdot (2 + w) = \omega \cdot \sup\{2 + n : n < \omega\} = \omega \cdot w = w^2 \]

(b)  
\[(\omega + 3) \cdot (\omega + 2) = (\omega + 3) \cdot (\omega + (\omega + 3) \cdot 2) \quad \text{(left distrib. law)} \]
\[= \sup\{(\omega + 3) \cdot n : n < \omega^2\} + (\omega + 3) \cdot (\omega + 3) \]
\[= \sup\{(\omega + 3) \cdot n + (\omega + 3) + \cdots + (\omega + 3) \cdot (\text{times}) : n < \omega^2\} + \omega + (\omega + 3) \cdot 3 \]
\[= \sup\{\omega \cdot n + 3 : n < \omega^2\} + \omega + \omega + 3 \]
\[= \omega + \omega + 2 \cdot 3 = \omega + \omega + 2 + 3 \]

because \[\omega^2 = \sup\{\omega \cdot n : n < \omega\}\le \sup\{\omega \cdot n + 3 : n < \omega\}\le \sup\{\omega \cdot (\omega + 1) : n < \omega^2\} = \omega^2 \]

3(a)  
\[k + \mu = |(\kappa \times \{0, 1\}) \cup (\mu \times 1)|, \quad k \cdot \mu = |k \times \mu| \]
\[k^\mu = |\mathcal{F}(\mu, k)| = |\text{set of all functions from } \mu \text{ to } k| \]

(b) We say that an ordinal \( \alpha \in K \) is odd if it can be expressed in the form \( \lambda + n \) and \( n \) is odd, where \( \lambda \) is a limit ordinal. We say \( \alpha \) is even if \( n \) is even.

Let \( K^{\text{odd}} = \text{set of all odd ordinals in } K \) and \( K^{\text{even}} = \text{set of all even ordinals in } K \).

Define \( f : (K \times \{0, 1\}) \cup (K \times \{1\}) \to K \) by
\[f(\langle \alpha, 0 \rangle) = \lambda + 2n \quad \text{if } \alpha = \lambda + n \]
\[f(\langle \alpha, 1 \rangle) = \lambda + 2n + 1 \quad \text{if } \alpha = \lambda + n + 1 \]

Then \( f \) is a bijection, provided \( K \) is infinite.

So \( K + K = |(K \times \{0, 1\}) \cup (K \times \{1\})| \]
\[= |K| \quad \text{because } K \subseteq (K \times \{0\}) \cup (K \times \{1\}) \]
\[= K \]

Hence \( K + K = K \) for all infinite cardinals \( K \).
4(a) Cantor-Bernstein Theorem: If \( A \preceq B \) and \( B \preceq A \) then \( A \cong B \). Here \( A \preceq B \) means that there is an injection from \( A \) to \( B \).

(b) Define \( f: \text{Seq}_F(\mathbb{N}) \rightarrow \mathbb{N} \) by
\[
f(<a_1, \ldots, a_n>) = 2^n p_1 a_1 p_2 a_2 \cdots p_n a_n\]
where \( p_1, p_2, \ldots, p_n, \ldots \) is the sequence of odd prime numbers in increasing order. Note when \( n = 0 \), \(<a_1, \ldots, a_n> = <> = \) the empty sequence, which has length 0. Now it follows from the Fundamental Theorem of Arithmetic that \( f \) is an injection. So \( \text{Seq}_F(\mathbb{N}) \preceq \mathbb{N} \). Also if we define \( g: \mathbb{N} \rightarrow \text{Seq}_F(\mathbb{N}) \) by \( g(k) = <k> \), then we see that \( g \) is also an injection. So \( \mathbb{N} \preceq \text{Seq}_F(\mathbb{N}) \).

So it follows from the Cantor-Bernstein Theorem that \( \text{Seq}_F(\mathbb{N}) \cong \mathbb{N} \).

5(a) Axiom of Choice (AC): If \( A \) is a set of disjoint non-empty sets then there is a set \( M \) which consists of one element of each member of \( A \).

Well-Ordering Principle (WOP): For each set \( A \), we can a relation \( \prec \) on \( A \) such that \( \langle A, \prec \rangle \) is a well-ordered set.

(b) WOP \( \Rightarrow \) AC: Suppose \( A \) is a set of disjoint non-empty sets. Then by WOP we can find a well-ordering \( \prec \) on \( UA \) because \( UA \) is a set. Now let each \( A \in A \) let \( f(A) = \) smallest element of \( A \) according to \( < \) and let's put \( M = \{ f(A) \mid A \in A \} \). Then \( M \) is a set because of the replacement axiom. Also \( M \) consists of one element of each member of \( A \). So WOP \( \Rightarrow \) AC.
6. (a) \( \text{cof}(\lambda) = \) smallest ordinal \( \theta \) such that there is a sequence \( \langle \alpha_\beta : \beta < \theta \rangle \) of ordinals in \( \lambda \) with \( \sup \{ \alpha_\beta : \beta < \theta \} = \lambda \).

A cardinal \( \kappa \) is strongly inaccessible if
(i) \( \kappa > \aleph_0 \), (ii) \( \kappa \) is regular, and
(iii) for each \( \mu < \kappa \), \( 2^\mu < \kappa \).

(b) Let \( \theta = \text{cof}(\lambda) \). Suppose \( \theta \) is not a cardinal. Then \( |\theta| < \theta \). Now by the definition of \( \text{cof}(\lambda) \), we can find a sequence \( \langle \alpha_\beta : \beta < \theta \rangle \) of ordinals in \( \lambda \) such that \( \sup \{ \alpha_\beta : \beta < \theta \} = \lambda \). Let \( \kappa = |\theta| \) and \( f : \kappa \to \theta \) be a bijection. Then \( \langle \alpha_{f(\gamma)} : \gamma < \kappa \rangle \) will be a sequence of ordinals in \( \lambda \) with \( \sup \{ \alpha_{f(\gamma)} : \gamma < \kappa \} = \lambda \). But this implies that \( \text{cof}(\lambda) \leq \kappa < \theta \) — a contradiction. Hence \( \theta \) must be a cardinal.