Cauchy-Euler Equations

The Cauchy-Euler equation has the form

\[ a_0 x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \ldots + a_{n-1} x y' + a_n y = b(x) \]

where \( a_0, a_1, \ldots, a_n \) are constants.

Each term contains \( x^k y^{(k)} \).

The transformation \( x = e^t \) reduces the equation to a linear O.D.E. with constant coefficient in the variable \( t \). Notice that we assume \( x > 0 \), and \( t = \ln x \).

Using chain rule, since \( y \) is function of \( t \) through \( x \):

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{dx} e^t = \frac{dy}{dx}
\]

then

\[
x \frac{dy}{dx} = \frac{dy}{dt}
\]

Now,

\[
\frac{d^2y}{dt^2} = \frac{d}{dt} \left[ \frac{dy}{dx} \right] = \frac{d}{dt} \left[ \frac{dx}{dt} \frac{dy}{dx} \right] = \frac{d}{dt} \frac{dx}{dt} \frac{dy}{dx} + x \frac{d}{dt} \left[ \frac{dy}{dx} \right] = e^t \frac{dy}{dx} + x \frac{d^2y}{dx^2} \frac{dx}{dt} = x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2}
\]

then

\[
x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}
\]

and

\[
\frac{d^3y}{dt^3} = \frac{d}{dt} \left[ \frac{d^2y}{dx^2} \right] = \frac{d}{dt} \left[ \frac{dx}{dt} \frac{d^2y}{dx^2} \right] = e^t \frac{dy}{dx} + x \frac{d^2y}{dx^2} \frac{dx}{dt} + 2e^2t \frac{d^2y}{dx^2} \frac{dx^2}{dt} + 2 \frac{d^3y}{dx^3} e^t =
\]

\[
x \frac{dy}{dx} + x^2 \frac{d^2y}{dx^2} + 2x \frac{d^2y}{dx^2} + x^3 \frac{d^3y}{dx^3} = \frac{dy}{dt} + 3x \frac{d^2y}{dx^2} \frac{dx}{dt} + x^3 \frac{d^3y}{dx^3} = \frac{dy}{dt} + 3 \frac{d^2y}{dt^2} - 3 \frac{dy}{dt} + x^3 \frac{d^3y}{dx^3}
\]

then

\[
x^3 \frac{d^3y}{dx^3} = \frac{d^3y}{dt^3} - 3 \frac{d^2y}{dt^2} + 2 \frac{dy}{dt}
\]

Following the model, you can compute the higher order derivatives.

Remark: For the second order equation, the transformation produces:

\[
ax^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = g(x), \ x > 0
\]

taking \( x = e^t \)

\[
a \frac{d^2y}{dt^2} + (b - a) \frac{dy}{dt} + cy = g(e^t)
\]
Example:

1) $x^2 y'' - 2xy' + 2y = x^3$, $x > 0$

Taking the transformation $x = e^t$ the equation reduces to:

$$y_{tt}'' + (-2 - 1) y_{t} + 2y = e^{3t}$$

or

$$y_{tt}'' - 3y_{t} + 2y = e^{3t}$$

Corresponding homogeneous equation: $y_{tt}'' - 3y_{t} + 2y = 0$

Characteristic equation: $r^2 - 3r + 2 = (r - 2)(r - 1) = 0$

Zeros are: $r_1 = 2$, and $r_2 = 1$

Fundamental set of solutions: $F = \{e^{2t}, e^t\}$

Complementary solution: $y_c(t) = c_1 e^{2t} + c_2 e^t$.

Non-homogeneous term is: $b(t) = e^{3t}$.

The UC set of $e^{3t}$ is $S_1 = \{e^{3t}\}$.

The candidate for particular solution is:

$$y_p(t) = Ae^{3t}$$

computing the derivatives,

$y_p'(x) = 3Ae^{3t}$

$y_p''(x) = 9Ae^{3t}$

substituting into the equation,

$$9Ae^{3t} - 9Ae^{3t} + 2Ae^{3t} = e^{3t}$$

then $A = \frac{1}{2}$

and

$$y_p(t) = \frac{1}{2} e^{3t}$$

$y(t) = c_1 e^{2t} + c_2 e^t + \frac{1}{2} e^{3t}$

since $x = e^t$, then the general solution of the original equation is:

$$y(x) = c_1 x^2 + c_2 x + \frac{1}{2} x^3$$

2) $x^2 y'' + 3xy' - 8y = \ln^3 x - \ln x$, $x > 0$

Taking the transformation $x = e^t$ the equation reduces to:

$$y_{tt}'' + (3 - 1) y_{t} - 8y = t^3 - t$$

or

$$y_{tt}'' + 2y_{t} - 8y = t^3 - t$$

Corresponding homogeneous equation: $y_{tt}'' + 2y_{t} - 8y = 0$

Characteristic equation: $r^2 + 2r - 8 = (r + 4)(r - 2) = 0$

Zeros are: $r_1 = 2$, and $r_2 = -4$

Fundamental set of solutions: $F = \{e^{2t}, e^{-4t}\}$

Complementary solution: $y_c(t) = c_1 e^{2t} + c_2 e^{-4t}$.

Non-homogeneous term is: $b(t) = t^3 - t$

The UC set of $t^3 - t$ is $S_1 = \{t^3, t^2, t, 1\}$.

The candidate for particular solution is:

$$y_p(t) = At^3 + Bt^2 + Ct + D$$

computing the derivatives,

$y_p'(x) = 3At^2 + 2Bt + C$
\[ y_p''(x) = 6At + 2B \]

substituting into the equation,
\[ 6At + 2B + 6A t^2 + 4Bt + 2C - 8At^3 - 8Bt^2 - 8Ct - 8D = t^3 - t \]
then
\[
\begin{align*}
-8A &= 1 & A &= -\frac{1}{8} \\
6A - 8B &= 0 & B &= -\frac{6}{64} = -\frac{3}{32} \\
6A + 4B - 8C &= -1 & C &= -\frac{1}{64} \\
2B + 2C - 8D &= 0 & D &= -\frac{7}{256}
\end{align*}
\]
and
\[
\begin{align*}
y_p(t) &= -\frac{1}{8} t^3 - 3\frac{1}{32} t^2 - 1\frac{1}{64} t - 7\frac{1}{256} \\
y(t) &= c_1 e^t + c_2 e^{2t} - 1\frac{1}{8} t^3 - 3\frac{1}{32} t^2 - 1\frac{1}{64} t - 7\frac{1}{256}
\end{align*}
\]
since \( x = e^t \) and \( t = \ln x \), then the general solution of the original equation is:
\[
y(x) = c_1 x^2 + c_2 x^3 - 1\frac{1}{8} \ln x - 3\frac{1}{32} \ln^2 x - 1\frac{1}{64} \ln x - 7\frac{1}{256}.
\]

3) \( x^3 y''' - 3x^2 y'' + 6xy' - 6y = \ln x, \; x > 0 \)

Taking the transformation \( x = e^t \) the equation reduces to:
\[
(y_t''' - 3y_t'' + 2y_t') - 3(y_t'' - y_t') + y_t'' - 6y = t
\]
or
\[
y_t''' - 6y_t''' + 11y_t' - 6y = t
\]
Corresponding homogeneous equation: \( y_t''' - 6y_t''' + 11y_t' - 6y = 0 \)
Characteristic equation: \( r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 3)(r - 2) = 0 \)
Zeros are: \( r_1 = 1, \; r_2 = 2, \; \text{and} \; r_3 = 3 \)
Fundamental set of solutions: \( F = \{ e^t, \; e^{2t}, \; e^{3t} \} \)
Complementary solution: \( y_c(t) = c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \).
Non-homogeneous term is: \( b(t) = t \)
The UC set of \( t \) is \( S_1 = \{ t, 1 \} \).
The candidate for particular solution is:
\[
y_p(t) = A + Bt
\]
computing the derivatives,
\[
y_p'(x) = B \\
y_p''(x) = 0 \\
y_p'''(x) = 0
\]
substituting into the equation,
\[
11B - 6A - 6Bt = t
\]
then
\[
\begin{align*}
-6B &= 1 & B &= -\frac{1}{6} \\
11B - 6A &= 0 & 11B = 6A = -1, & B = -\frac{1}{11}
\end{align*}
\]
and
\[
\begin{align*}
y_p(t) &= -\frac{1}{6} - \frac{1}{11} t \\
y(t) &= c_1 e^t + c_2 e^{2t} + c_3 e^{3t} - \frac{1}{6} - \frac{1}{11} t
\end{align*}
\]
since \( x = e^t \) and \( t = \ln x \), then the general solution of the original equation is:
\[
y(x) = c_1 x + c_2 x^2 + c_3 x^3 - \frac{1}{6} - \frac{1}{11} \ln x.
\]
4) \((x + 2)^2 y'' - (x + 2) y' + y = 3x + 2\)

Take the transformation \(x + 2 = e^t\), then \(t = \ln(x + 2)\) and \(e^t - 2 = x\)

\[
\frac{dt}{dx} = \frac{1}{x + 2}
\]

\[
\frac{dy}{dx} = \frac{dy}{dt} \frac{dt}{dx} = \frac{1}{x + 2} \frac{dy}{dt} \text{ or } (x + 2) \frac{dy}{dt} = \frac{dy}{dx}
\]

\[
(x + 2)^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}
\]

\(b(x) = 3x + 2, \quad b(t) = 3(e^t - 2) + 2 = 3e^t - 4.\)

The transformed equation is:

\[y_i'' - 2y_i' + y = 3e^t - 4\]

Corresponding homogeneous equation:

\[y_i'' - 2y_i' + y = 0\]

Characteristic equation: \(r^2 - 2r + 1 = (r - 1)^2 = 0\)

Zeros are: \(r_1 = 1\) double zero

Fundamental set of solutions: \(F = \{e^t, te^t\}\)

Complementary solution: \(y_c(t) = c_1e^t + c_2te^t.\)

Non-homogeneous term is: \(b(t) = 3e^t - 4\)

The UC set of \(e^t\) is \(S_1 = \{e^t\}\), and the UC set of 1 is \(S_2 = \{1\}\)

Since \(S_1\) contains elements of the Fundamental Set of solutions, so replace \(S_1\) by \(S_1^* = \{t^2e^t\}\)

The candidate for particular solution is:

\[y_p(t) = At^2e^t + B\]

computing the derivatives,

\[y_p'(x) = 2At^2e^t + At^2e^t\]

\[y_p''(x) = 2Ae^t + 4Ate^t + At^2e^t\]

substituting into the equation,

\[2A + 4Ae^t + At^2e^t - 2Ate^t + At^2e^t + B = 2Ae^t + B = 3e^t - 4\]

then \(2A = 3, \quad A = 3/2\) and \(B = -4\)

and

\[y_p(t) = 3/2 t^2e^t - 4\]

\[y(t) = c_1e^t + c_2te^t + 3/2 t^2e^t - 4\]

since \(x + 2 = e^t\) and \(t = \ln(x + 2)\), then the general solution of the original equation is:

\[y(x) = c_1(x + 2) + c_2(x + 2) \ln(x + 2) + 3/2 (x + 2)\ln^2(x + 2) - 4.\]