**Introduction and Review of Power Series**

**Definition:** A power series in powers of \( x - a \) is an infinite series of the form

\[
\sum_{n=0}^{\infty} c_n (x - a)^n = c_0 + c_1 (x - a) + c_2 (x - a)^2 + \ldots + c_n (x - a)^n + \ldots
\]

If \( a = 0 \), this is a power series in \( x \):

\[
\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots
\]

**Definition:** A power series \( \sum_{n=0}^{\infty} c_n (x - a)^n \) is said to converge at a point \( x \) if

\[
\lim_{p \to \infty} \sum_{n=0}^{p} c_n (x - a)^n
\]

exists for that \( x \).

This series certainly converges for \( x = a \). It may converge for all values of \( x \), or it may converge for some values of \( x \) and not for others.

**Definition:** A power series \( \sum_{n=0}^{\infty} c_n (x - a)^n \) is said to converge absolutely at a point \( x \) if

\[
\sum_{n=0}^{\infty} |c_n (x - a)^n| = \sum_{n=0}^{\infty} |c_n| |x - a|^n
\]

converges.

**Remark:** A series that converges absolutely also converges, but the converse is not necessarily true.

**The Ratio Test**

One of the most useful tests for absolute convergence of a power series is the ratio test. If \( c_n \neq 0 \), and if for a fixed value of \( x \),

\[
\lim_{n \to \infty} \left| \frac{c_{n+1} (x - a)^{n+1}}{c_n (x - a)^n} \right| = |x - a| \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right| = |x - a| L
\]

then the power series converges absolutely at that value of \( x \) if \( |x - a| L < 1 \) and diverges if \( |x - a| L > 1 \). If \( |x - a| L = 1 \), the test is inconclusive.

**Example:**

Find the values of \( x \) where the power series converges

\[
\sum_{n=0}^{\infty} (-1)^{n+1} n (x - 2)^n
\]

Applying the ratio test, we have
According to the ratio test the series converges absolutely if \(|x - 2| < 1\), or \(1 < x < 3\), and diverges if \(|x - 2| > 1\).

The values of \(x\) corresponding to \(|x - 2| = 1\) are \(x = 1\) and \(x = 3\). The series diverges for both values since the \(n\)th term of the series does not approach zero when \(n \to \infty\).

**Remark**: If the power series \(\sum_{n=0}^{\infty} c_n (x-a)^n\) converges at \(x = b\), it converges absolutely for \(|x - a| < |b - a|\). If the series diverges for \(x = b\), it diverges for \(|x - a| > |b - a|\).

**Radius of Convergence**

There is a nonnegative number \(\rho\), called **radius of convergence**, such that

\[\sum_{n=0}^{\infty} c_n (x-a)^n\] converges absolutely for \(|x - a| < \rho\) and diverges for \(|x - a| > \rho\).

For the series that only converges at \(x = a\), we define \(\rho = 0\).
If \(\rho > 0\), the interval \((a - \rho, a + \rho)\) is called **interval of convergence**.

The series may either converge or diverge when \(|x - a| = \rho\).

**Example**:

Determine the radius of convergence of the series

\[\sum_{n=0}^{\infty} \frac{(x + 1)^n}{n2^n}\]

We apply the ratio test

\[\lim_{n \to \infty} \left| \frac{(x + 1)^{n+1}}{(n + 1)2^{n+1}} \cdot \frac{n2^n}{(x + 1)^n} \right| = \frac{|x + 1|}{2} \lim_{n \to \infty} \frac{n}{n + 1} = \frac{|x + 1|}{2}\]

the series converges absolutely if \(|x + 1| < 2\) or \(-3 < x < 1\), and diverges for \(|x + 1| > 1\).

The radius of convergence is 2.

At \(x = 1\), the series becomes the harmonic series \(\sum_{n=0}^{\infty} \frac{1}{n}\) that diverges; at \(x = -3\), we have

\[\sum_{n=0}^{\infty} \frac{(-1)^n}{n}\] which converges, but does not converge absolutely.
**Operation with Power Series**

Given the power series \( \sum_{n=0}^{\infty} a_n (x-a)^n \) and \( \sum_{n=0}^{\infty} b_n (x-a)^n \) that converges to \( f(x) \) and \( g(x) \), respectively, for \(|x-a| < \rho\), \( \rho > 0 \).

1) The series can be add or subtract termwise, and

\[
f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x-a)^n
\]

the resulting series converges at least for \(|x-a| < \rho\).

**Example:** Add

\[
\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n} + \sum_{n=0}^{\infty} \frac{(x-3)^n}{(n+1)^n} = \sum_{n=0}^{\infty} \left( \frac{1}{2^n} + \frac{1}{(n+1)^n} \right) (x+3)^n
\]

2) The series can be formally multiply, and

\[
f(x)g(x) = \left[ \sum_{n=0}^{\infty} a_n (x-a)^n \right] \left[ \sum_{n=0}^{\infty} b_n (x-a)^n \right] = \sum_{n=0}^{\infty} c_n (x-a)^n
\]

where \( c_n = a_0b_n + a_1b_{n-1} + \ldots + a_nb_0 \).

the resulting series converges at least for \(|x-a| < \rho\).

**Example:** Multiply

\[
e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots \quad \text{and} \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \ldots
\]

\[
e^x \cos x = \left[ (1-\frac{x^2}{2} + \frac{x^4}{24} + \ldots) - \frac{x^2}{2} (1-\frac{x^2}{2} + \frac{x^4}{24} + \ldots) + \frac{x^4}{24} (1-\frac{x^2}{2} + \frac{x^4}{24} + \ldots) + \ldots \right] + \ldots
\]

\[
= 1 + x + (-\frac{x^2}{2} + \frac{x^2}{2}) + (\frac{x^3}{6} - \frac{x^3}{2}) + (\frac{x^4}{24} - \frac{x^4}{4} + \frac{x^4}{24}) + \ldots = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \ldots
\]

3) The quotient of two power series can be computed by long division, if \( g(a) \neq 0 \).

**Example:**

Divide the power series \( \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \ldots \) by

\( \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \ldots \)
The Taylor Series Expansion

If \( f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n \) is continuous and has derivatives of all orders for \( |x - a| < \rho \), then \( f' \), \( f'' \), \( \ldots \), can be computed by differentiating the series termwise; that is,

\[
f'(x) = c_1 + 2c_2(x - a) + \ldots + nc_n(x - a)^{n-1} + \ldots = \sum_{n=1}^{\infty} nc_n(x - a)^{n-1}
\]

\[
f''(x) = 2c_2 + 6c_3(x - a) + \ldots + n(n-1)c_n(x - a)^{n-2} + \ldots = \sum_{n=1}^{\infty} n(n-1)c_n(x - a)^{n-2}
\]

and so forth, each of the series converges absolutely for \( |x - a| < \rho \).

Notice that the value of \( c_n \) is given by

\[
c_n = \frac{f^{(n)}(a)}{n!}
\]

so

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n-1)}(a)}{n!} (x - a)^n
\]

is called the Taylor series expansion of \( f \) about \( x = a \).

If \( x = 0 \), then we have the MacLaurin series:
Some Important MacLaurin Series:

<table>
<thead>
<tr>
<th>MacLaurin Series</th>
<th>Interval of Convergence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{x} )</td>
<td>(-1 &lt; x &lt; 1)</td>
</tr>
<tr>
<td>( \frac{1}{x^2 + 1} )</td>
<td>(-1 &lt; x &lt; 1)</td>
</tr>
<tr>
<td>( e^x )</td>
<td>(-\infty &lt; x &lt; \infty)</td>
</tr>
<tr>
<td>( e^{-x} )</td>
<td>(-\infty &lt; x &lt; \infty)</td>
</tr>
<tr>
<td>( \sin x )</td>
<td>(-\infty &lt; x &lt; \infty)</td>
</tr>
<tr>
<td>( \cos x )</td>
<td>(-\infty &lt; x &lt; \infty)</td>
</tr>
<tr>
<td>( \arctan x )</td>
<td>(-\infty &lt; x &lt; \infty)</td>
</tr>
</tbody>
</table>

**Definition**: A function \( f(x) \) that has a Taylor series expansion about \( x = a \) with radius of convergence \( \rho > 0 \), is said to be **analytic** at \( a \).

**Example**:
1) Polynomial functions are analytic everywhere.
2) The functions in the MacLaurin series expansion are analytic about \( x = 0 \).
3) A rational functions \( P(x)/Q(x) \) is analytic everywhere except at those values where \( Q(x) = 0 \).

**Remark**: If \( \sum_{n=0}^{\infty} c_n (x-a)^n = \sum_{n=0}^{\infty} d_n (x-a)^n \) for all \( x \) in some open interval centered at \( a \), then \( c_n = d_n \) for all \( n \). In particular, if \( \sum_{n=0}^{\infty} c_n (x-a)^n = 0 \) for each such \( x \), then \( c_n = 0 \) for all \( n \).

**Shift of Index of Summation**
The index of summation in an infinite series is a dummy parameter just as the integration variable in a definite integral is a dummy variable. Thus it is immaterial which letter is used for the index of summation.

**Example:**

$$
\sum_{n=0}^{\infty} \frac{2^n x^n}{n!} = \sum_{k=0}^{\infty} \frac{2^k x^k}{k!} = \sum_{p=0}^{\infty} \frac{2^p x^p}{p!}
$$

We can also make changes in the index of summation, we can increase the index of summation and decrease the starting point or we can decrease the index and increase the starting point.

**Example:**

1) Write the series \( \sum_{n=2}^{\infty} c_n x^n \) as a series whose first term corresponds to \( n = 0 \) rather than \( n = 2 \).

Since we have to decrease the starting point by 2, we have to increase the index of summation by 2.

Let \( p = n - 2 \), then \( n = p + 2 \) and \( n = 0 \), implies \( p = 2 \)

\[
\sum_{n=2}^{\infty} c_n x^n = \sum_{p=0}^{\infty} c_{p+2} x^{p+2} = \sum_{n=0}^{\infty} c_{n+2} x^{n+2}
\]

we can replace the dummy index \( p \) back to \( n \).

2) Write the series \( \sum_{n=2}^{\infty} (n + 1) c_n x^{n+2} \) as a series whose generic term involves \( x^n \) rather than \( x^{n+2} \).

Since we have to decrease the index of summation by 2, we have to increase the starting point by 2.

Let \( p = n + 2 \), then \( n = p - 2 \), and \( n = 2 \), implies \( p = 4 \),

\[
\sum_{n=2}^{\infty} (n + 1) c_n x^{n+2} = \sum_{p=4}^{\infty} (p - 2 + 1) c_{p-2} x^p = \sum_{n=4}^{\infty} (p - 1) c_{n-1} x^n
\]