Separable Equations

**Definition** An O.D.E. of the form

\[ F(x) G(y) \, dx + f(x) g(y) \, dy = 0 \]

is called an equation with variable separable or a separable equation.

**Example:**
The O.D.E.

\[ \frac{(x^3 - 3) \, y^2}{F(x)} \, dx - \frac{x \, (y + 1)}{f(x)} \, dy = 0 \]

is separable.

In general a separable equation is non-exact but it has an obvious integrating factor,

\[ p(x,y) = \frac{1}{f(x)G(y)} \]

when we multiply the equation by \( p(x,y) \), we get the essential equivalent exact equation

\[ \frac{F(x)}{f(x)} \, dx + \frac{g(y)}{G(y)} \, dy = 0 \]

since

\[ \frac{\partial}{\partial y} \left( \frac{F(x)}{f(x)} \right) = 0 = \frac{\partial}{\partial x} \left( \frac{g(y)}{G(y)} \right). \]

Since the first term of the equation depends on \( x \) only, we can integrate it with respect to \( x \). Since the second term depends on \( y \) only, we can integrate it with respect to \( y \).

\[ \int \frac{F(x)}{f(x)} \, dx + \int \frac{g(y)}{G(y)} \, dy = c \]

and we obtain the one-parameter family of solutions of the separable equation.

**Remark:** Notice that we must separate the variables before integration.

**Remark:** To separate the variables, we divide by \( f(x)G(y) \). Since division by zero is not allowed, we must assume that neither \( f(x) = 0 \) or \( G(y) = 0 \).

This assumption may produce the lost or gain of solutions, so we have to investigate these possibilities.

Regarding \( y \) as the dependent variable, we consider the values of \( y \) that make \( G(y) = 0 \). Let’s say \( G(y) = 0 \) when \( y = y_0 \), then the constant function \( h(y) = y_0 \) is a solution of the original equation

\[ F(x)G(y) + f(x)g(y) \frac{dy}{dx} = 0. \]

Therefore, we must check that the constant function \( h(y) = y_0 \) is part of the one-parameter family of solutions of the separated equation. This means, there is a value of the parameter \( c \), that produces \( y = y_0 \).

If that is not possible, then the constant function must be added to the family of solutions.

**Example:** Solve the O.D.E.
1) \( \frac{x^2 + 1}{x} \, dy + y \, dx = 0 \)

This equation is separable, we separate the variables dividing by \( y \frac{x^2 + 1}{x} \), to obtain

\[
\frac{x}{x^2 + 1} \, dx + \frac{1}{y} \, dy = 0
\]

integrating each term, we get

\[
\int \frac{x}{x^2 + 1} \, dx + \int \frac{1}{y} \, dy = c
\]

\[
\frac{1}{2} \ln |x^2 + 1| + \ln |y| = c
\]

\[
\ln \left( |x^2 + 1|^{\frac{1}{2}} |y| \right) = c \quad \text{or} \quad |x^2 + 1|^{\frac{1}{2}} |y| = e^c \quad \text{or} \quad y = k(x^2 + 1)^{\frac{1}{2}}
\]

Notice that \( G(y) = y = 0 \) is a solution of the equation and it is included in the one-parameter family of solution, it is obtained by making \( k = 0 \).

Then for any choice of \( k \), \( y = k(x^2 + 1)^{\frac{1}{2}} \) is a solution of the separable equation.

2) \( xy^4 \, dx + (y^2 + 2)e^{-3x} \, dy = 0 \)

The equation is separable, we separate the variables dividing by \( e^{-3x}y^4 \), to obtain

\[
x e^{3x} \, dx + \frac{y^2 + 2}{y^4} \, dy = 0
\]

integrating each term, we get

\[
\int xe^{3x} \, dx + \int \frac{1}{y^2} \, dy + \int \frac{2}{y^4} \, dy = c
\]

using integration by parts

\[
\frac{1}{3} xe^{3x} - \frac{1}{9} e^{3x} - y^{-1} - \frac{2}{3} y^{-3} = c
\]

\[
e^{3x}(3x - 1) = \frac{9}{y} + \frac{6}{y^3} + m
\]

\[
\left[ e^{3x}(3x - 1) + k \right] y^3 = 9y^2 + 6
\]

Notice that \( G(y) = y^4 = 0 \), produces the constant function \( y = 0 \) that is a solution of the original equation, but there is no value of constant \( k \) such that the function belongs to the family of solutions. So we lost such solution in the process of separating variables.

Therefore, the solution of the equation is the family \( \left[ e^{3x}(3x - 1) + k \right] y^3 = 9y^2 + 6 \) and the constant function \( y = 0 \).

3) Solve the I.V.P.
\[
\begin{align*}
\begin{cases}
\frac{dy}{dx} = y^2 - 4 \\
y(0) = -2
\end{cases}
\end{align*}
\]

We separate the variable dividing by \( y^2 - 4 \),
\[
\frac{dy}{y^2 - 4} = \frac{dy}{y - 2} + \frac{dy}{y + 2} = 0
\]

Using partial fractions:
\[
\frac{1}{y^2 - 4} = \frac{A}{y - 2} + \frac{B}{y + 2} = \frac{Ay + 2A + By - 2B}{y^2 - 4}
\]

Then \((A + B)y + (2A - 2B) = 1\) or \(A + B = 0\) and \(2A - 2B = 1\), \( A = \frac{1}{4}\) and \(B = -\frac{1}{4}\).

\[
\int \frac{1}{4} \frac{dy}{y - 2} + \int -\frac{1}{4} \frac{dy}{y + 2} - \int dx = c
\]

then,
\[
\frac{1}{4} \ln |y - 2| - \frac{1}{4} \ln |y + 2| = x + c
\]

\[
\ln \left| \frac{y - 2}{y + 2} \right| = 4x + m
\]

\[
\left| \frac{y - 2}{y + 2} \right| = e^m e^{4x}
\]

we can simplify it as
\[
\frac{y - 2}{y + 2} = ke^{4x} \quad \text{or} \quad y - 2 = k e^{4x} + 2 e^{4x} \quad \text{or} \quad y - k e^{4x} = 2 + 2 e^{4x} \quad \text{or} \quad y = 2 \frac{1 + ke^{4x}}{1 - ke^{4x}}
\]

If we try to solve the I.V.P., we substitute \(x = 0\) and \(y = -2\), to obtain
\[
-2 = 2 \frac{1 + k}{1 - k} \quad \text{or} \quad -1 + k = 1 + k \quad \text{or} \quad -1 = 1, \quad \text{contradiction}.
\]

We reach a contradiction because the one-parameter family of solutions does not contain the constant function \(y = 2\) obtained from \(G(y) = y^2 - 4 = 0\) (solutions: \(y = \pm 2\)).

Notice that making \(k = 0\) we get \(y = 2\), but there is no finite value of \(k\) which will yield the solution \(y = -2\).

Therefore, the solution of the I.V.P. is the constant function \(y(x) = -2\).