Series Solutions Near an Ordinary Point

We are considering methods of solving second order linear equations when the coefficients are functions of the independent variable. We consider the second order linear homogeneous equation

$$P(x) \frac{d^2y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = 0 \quad (1)$$

since the procedure for the non-homogeneous equation is similar. Many problems in mathematical physics lead to equations of this form having polynomial coefficients; examples include the Bessel equation

$$x^2y'' + xy' + (x^2 - a^2)y = 0$$

where \(a\) is a constant, and the Legendre equation

$$(1 - x^2)y'' - 2xy' + c(c + 1)y = 0$$

where \(c\) is a constant.

For this reason, as well as to simplify the algebraic computations, we primarily consider the case in which \(P, Q,\) and \(R\) are polynomials. However, we will see that the method can be applied when \(P, Q,\) and \(R\) are analytic functions. For the time being, we assume \(P, Q,\) and \(R\) are polynomials and they do not have common factors. Suppose that we wish to solve equation (1) in a neighborhood of a point \(a.\) The solution of equation (1) in an interval containing point \(a\) is closely related with the behavior of \(P(x)\) in the interval.

**Definition:** Given the equation

$$P(x) \frac{d^2y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = 0$$

the equation

$$\frac{d^2y}{dx^2} + \frac{Q(x)}{P(x)} \frac{dy}{dx} + \frac{R(x)}{P(x)}y = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} + p(x) \frac{dy}{dx} + q(x)y = 0$$

where \(p(x) = \frac{Q(x)}{P(x)}\) and \(q(x) = \frac{R(x)}{P(x)}\)

is called the equivalent normalized form of equation (1).

**Definition:** The point \(a\) is called an ordinary point of equation (1) if both of the functions \(p(x)\) and \(q(x)\) in the equivalent normalized form, are analytic functions at the point \(a.\) If either or both of these functions are not analytic at \(a,\) then the point \(a\) is a singular point of equation (1).

**Examples:**
1) Given \(y'' + xy' + (x^2 + 2) = 0\)
   since \(p(x) = x\) and \(q(x) = x^2 + 2\) are polynomials, then they are analytic functions everywhere, then every real number is an ordinary point.
2) Given \((x^2 - 1)y'' + xy' + \frac{1}{x}y = 0\) \((x^2 - 1)\)

the equivalent normalized equation is

\[
y'' + \frac{x}{x^2 - 1}y' + \frac{1}{x(x^2 - 1)}y = 0
\]

where \(p(x) = \frac{x}{x^2 - 1}\) and \(q(x) = \frac{1}{x(x^2 - 1)}\)

then the points 1, -1, and 0 are singular points of the equation, any other real number is an ordinary point of the equation.

**Theorem:** If \(a\) is an ordinary point of the differential equation (1),

then \(p(x) = \frac{Q(x)}{P(x)}\) and \(q(x) = \frac{R(x)}{P(x)}\) are analytic at \(a\), and there are two nontrivial linearly independent power series solutions of equation (1) of the form

\[
\sum_{n=0}^{\infty} c_n(x - a)^n, \quad \sum_{n=0}^{\infty} d_n(x - a)^n
\]

Furthermore, the radius of convergence for each of the series solutions is at least as large as the minimum of the radii of convergence of the series for \(p(x)\) and \(q(x)\).

**Example:**

1) The point \(a = 0\) is an ordinary point of the equation

\[
y''' + (x^3 + 1) y' - xy = 0
\]

then, the differential equation has a solution in the form \(\sum_{n=0}^{\infty} c_n x^n\) whose radius of convergence is \(\rho = \infty\).

2) The differential equation

\[
(x^2 - 4)y''' + 3xy' + y = 0
\]

has two singular points 2 and \(-2\).

Then the differential equation has a solution in the form \(\sum_{n=0}^{\infty} c_n x^n\) whose radius of convergence is at least \(\rho = 2\).

**The Method of Solution**

We assume that \(a\) is an ordinary point of the equation (1), so it has a series solution near \(a\) of the form \(\sum_{n=0}^{\infty} c_n (x - a)^n\) that converges for \(|x - a| < \rho\).

We want to determine the coefficients \(c_0, c_1, c_2, \ldots\) in the series solution. Let’s differentiate the series termwise twice.
\[
y(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots
\]

\[
y'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} = c_1 + 2c_2 (x-a) + 3c_3 (x-a)^2 + \ldots
\]

\[
y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n (x-a)^{n-2} = 2c_2 + 6c_3 (x-a) + 12c_4 (x-a)^2 + \ldots
\]

Substituting in the differential equation (1), we get
\[
K_0 + K_1(x-a) + K_2(x-a)^2 + \ldots + K_n(x-a)^n + \ldots = 0 \quad (4)
\]

where \(K_n\) is a function of certain coefficients \(c_i\).

In order that series (4) be valid for all \(x\) in some interval \(|x-a| < \rho\), we must have
\[
K_0 = K_1 = \ldots = K_n = \ldots = 0
\]

This leads to a set of conditions that the coefficients \(c_n\) must satisfy.

**Example:**

1) Solve \((x^2 - 4)y'' + 3xy' + y = 0\) in powers of \(x\).

The only singular points of the equation are 2 and -2, so we want to find the series solution of the equation near zero. The series will have radius of convergence at least 2.

So \(y(x) = \sum_{n=0}^{\infty} c_n x^n, \quad y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}, \quad y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}\)

Substituting the series in the equation
\[
x^2 \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 4 \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + 3x \sum_{n=1}^{\infty} nc_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0
\]

\[
\sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=2}^{\infty} 4n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} 3nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0
\]

we shift the index of summation in the second series by 2, replacing \(n\) with \(n+2\) and using the initial value \(n = 0\).

Since we want to express everything in only one summation sign, we have to start the summation at \(n = 2\) in every series,

\[
\sum_{n=2}^{\infty} n(n-1)c_n x^n - \sum_{n=0}^{\infty} 4(n+2)(n+1)c_{n+2} x^n + \sum_{n=1}^{\infty} 3nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0
\]

\[
\sum_{n=2}^{\infty} n(n-1)c_n x^n - 4(2)(1)c_2 - 4(3)(2)c_3 x - \sum_{n=2}^{\infty} 4(n+2)(n+1)c_n x^n + 3c_1 x + \sum_{n=2}^{\infty} 3nc_n x^n = 0
\]

\[c_0 + c_1 x + \sum_{n=2}^{\infty} c_n x^n = 0\]
Substituting the series in the equation near zero. The series converges for all \( x \).

There are no singular points of the equation, so we want to find the series solution of the
2) Solve \( y'' + xy' + (x^2 + 2)y = 0 \) in powers of \( x \).

This last condition is called a **recurrence formula**, we can express each \( c_{n+2} \) in terms of a
previous coefficient \( c_n \).

\[
\begin{align*}
\text{n} = 2, & \quad c_4 = \frac{3c_2}{4 \cdot 4} = \frac{3c_0}{128} \\
\text{n} = 3, & \quad c_5 = \frac{4c_3}{4 \cdot 5} = \frac{2 \cdot 4c_1}{30} \\
\text{n} = 4, & \quad c_6 = \frac{5c_4}{4 \cdot 6} = \frac{3 \cdot 5c_0}{1024} \\
\text{n} = 5, & \quad c_7 = \frac{6c_5}{4 \cdot 7} = \frac{2 \cdot 4 \cdot 6c_1}{140}
\end{align*}
\]

Notice that each even coefficient is expressed in terms of \( c_0 \) and each odd coefficient is
expressed in terms of \( c_1 \).

Then, the general solution is:

\[
y(x) = c_0 \left( 1 + \frac{1}{8} x^2 + \frac{3}{128} x^4 + \frac{5}{1024} x^6 + \ldots \right) + c_1 \left( x + \frac{1}{6} x^3 + \frac{1}{30} x^5 + \frac{1}{140} x^7 + \ldots \right)
\]

2) Solve \( y'' + xy' + (x^2 + 2)y = 0 \) in powers of \( x \).

There are no singular points of the equation, so we want to find the series solution of the
equation near zero. The series converges for all \( x \).

So \( y(x) = \sum_{n=0}^{\infty} c_n x^n \), \( y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1} \), \( y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \)

Substituting the series in the equation

\[
\begin{align*}
\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x \sum_{n=1}^{\infty} nc_n x^{n-1} + x^2 \sum_{n=0}^{\infty} c_n x^n + 2 \sum_{n=0}^{\infty} c_n x^n = \\
\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^{n+2} + \sum_{n=0}^{\infty} 2c_n x^n = \\
\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0
\end{align*}
\]
We shift the index of summation in the first series by 2, replacing \( n \) with \( n + 2 \) and using the initial value \( n = 0 \). We shift the index of summation in the third series by -2, replacing \( n \) by \( n - 2 \) and using the initial value \( n = 2 \).

Since we want to express everything in only one summation sign, we have to start the summation at \( n = 2 \) in every series,

\[
\sum_{n=0}^{\infty} (n + 2)(n + 1)c_{n+2}x^n + \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=2}^{\infty} c_{n-2}x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0
\]

\[
2c_2 + 6c_3 + \sum_{n=2}^{\infty} (n + 2)(n + 1)c_{n+2}x^n + c_1x + \sum_{n=2}^{\infty} nc_n x^n + 3c_1x + \sum_{n=2}^{\infty} c_{n-2}x^n +
\]

\[
2c_0 + 2c_1x + \sum_{n=2}^{\infty} c_n x^n = 0
\]

\[
(2c_0 + 2c_2) + (3c_1 + 6c_3)x - \sum_{n=2}^{\infty} [(n + 2)(n + 1)c_{n+2} + (n + 2)c_n + c_{n-1}]x^n = 0
\]

Now

\[
2c_0 + 2c_2 = 0, \text{ then } c_2 = -c_0
\]

\[
3c_1 + 6c_3 = 0, \text{ then } c_3 = -\frac{1}{2}c_1
\]

\[
(n + 2)(n + 1)c_{n+2} + (n + 2)c_n + c_{n-2} = 0, \text{ then } c_{n+2} = -\frac{(n+2)c_n + c_{n-2}}{(n+1)(n+2)}, n \geq 2
\]

This last condition is called a recurrence formula, we can express each \( c_{n+2} \) in terms of a previous coefficient \( c_{n-2} \) and \( c_n \).

\[
n = 2, c_4 = -\frac{4c_2 + c_0}{12} = -\frac{-4c_0 + c_0}{12} = \frac{3c_0}{12} = \frac{1}{4}c_0
\]

\[
n = 3, c_5 = -\frac{5c_3 + c_1}{20} = -\frac{5(-\frac{1}{2}c_1 + c_1)}{20} = \frac{3c_1}{40}
\]

\[
n = 4, c_6 = -\frac{6c_4 + c_2}{5 \cdot 6} = -\frac{6(\frac{1}{4}c_0 - c_0)}{30} = -\frac{c_0}{60}
\]

\[
n = 5, c_7 = -\frac{7c_5 + c_3}{6 \cdot 7} = -\frac{7(\frac{3}{40}c_1 - \frac{1}{2}c_1)}{42} = -\frac{c_1}{1680}
\]

Notice that each even coefficient is expressed in terms of \( c_0 \) and each odd coefficient is expressed in terms of \( c_1 \).

Then, the general solution is:

\[
y(x) = c_0 \left(1 - x^2 + \frac{1}{4}x^4 - \frac{1}{60}x^6 + \ldots \right) + c_1 \left(x - \frac{1}{2}x^3 + \frac{3}{4}x^5 - \frac{1}{1680}x^7 + \ldots \right)
\]

3) Solve \( y'' - xy' - x^2y = 0 \) in powers of \( x \).

There are no singular points of the equation, so we want to find the series solution of the equation near zero. The series converges for all \( x \).

So \( y(x) = \sum_{n=0}^{\infty} c_n x^n, \ y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}, \ y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \).
Substituting the series in the equation
\[
\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x \sum_{n=1}^{\infty} nc_n x^{n-1} - x^2 \sum_{n=0}^{\infty} c_n x^n =
\]
\[
\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=2}^{\infty} c_n x^{n+2} =
\]
\[
\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=2}^{\infty} c_{n-2} x^n = 0
\]
We shift the index of summation in the first series by 2, replacing \(n\) with \(n + 2\) and using the initial value \(n = 0\). We shift the index of summation in the third series by -2, replacing \(n\) by \(n - 2\) and using the initial value \(n = 2\). Since we want to express everything in only one summation, we have to start the summation at \(n = 2\) in every series,
\[
\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=2}^{\infty} c_{n-2} x^n =
\]
\[
2c_2 + 6c_3 x + \sum_{n=2}^{\infty} (n+2)(n+1)c_{n+2} x^n - c_1 x - \sum_{n=2}^{\infty} nc_n x^n + \sum_{n=2}^{\infty} c_{n-2} x^n =
\]
\[
2c_2 + (6c_3 - c_1) x + \sum_{n=2}^{\infty} [(n+2)(n+1)c_{n+2} - nc_n - c_{n-2}] x^n = 0
\]
Now
\[
2c_2 = 0, \text{ then } c_2 = 0
\]
\[
6c_3 - c_1 = 0, \text{ then } c_3 = -1/6 c_1
\]
\[
c_{n+2} = \frac{nc_n + c_{n-2}}{(n+2)(n+1)}, \quad n \geq 2
\]
This last condition is called a **recurrence formula**, we can express each \(c_{n+2}\) in terms of a previous coefficient \(c_{n-2}\) and \(c_n\).

\[
n = 2, \quad c_4 = \frac{2c_2 + c_0}{12} = \frac{c_0}{12}
\]
\[
n = 3, \quad c_5 = \frac{3c_3 + c_1}{20} = \frac{\frac{1}{2}c_1 + c_1}{20} = \frac{3c_1}{40}
\]
\[
n = 4, \quad c_6 = \frac{4c_4 + c_2}{30} = \frac{4\left(\frac{1}{12}\right)c_0}{30} = \frac{c_0}{90}
\]
\[
n = 5, \quad c_7 = \frac{5c_5 + c_3}{42} = \frac{\frac{3}{40}c_1 + \frac{1}{6}c_1}{42} = \frac{13c_1}{1080}
\]

Notice that each even coefficient is expressed in terms of \(c_0\) and each odd coefficient is expressed in terms of \(c_1\).

Then, the general solution is:
\[ y(x) = c_0 \left( 1 + \frac{1}{12} x^4 + \frac{1}{90} x^6 + \ldots \right) + c_1 \left( x + \frac{1}{6} x^3 + \frac{3}{40} x^5 + \frac{13}{1080} x^7 + \ldots \right) \]

4) Solve \((x^2 + 1)y^{\prime\prime} - y^{\prime} + y = 0\) in powers of \(x\).
There are no singular points of the equation, so we want to find the series solution of the equation near zero. The series converges for all \(x\).

So \( y(x) = \sum_{n=0}^{\infty} c_n x^n \), \( y(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \), \( y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \)

Substituting the series in the equation
\[
x^2 \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n =
\]

\[
\sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0
\]

we shift the index of summation in the second series by 2, replacing \(n\) with \(n+2\) and using the initial value \(n = 0\), and we shift the index of summation in the third series by 1, replacing \(n\) with \(n+1\) and using the initial value \(n = 0\).

Since we want to express everything in only one summation sign, we have to start the summation at \(n = 2\) in every series,
\[
\sum_{n=2}^{\infty} n(n-1) c_n x^n + \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} c_n x^n =
\]

\[
2c_2 + (3)(2) c_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) c_{n+2} x^n + \sum_{n=2}^{\infty} n(n-1) c_n x^n - c_1 - 2c_2 x + \sum_{n=2}^{\infty} (n+1) c_{n+1} x^n +
\]

\[
c_0 + c_1 x + \sum_{n=2}^{\infty} c_n x^n = 0
\]

\[
(2c_2 - c_1 + c_0) + (6c_3 - 2c_2 + c_1) x + \sum_{n=2}^{\infty} \left[ (n+2)(n+1)c_{n+2} + n(n-1)c_n - (n+1)c_{n+1} + c_n \right] x^n =
\]

\[
(2c_2 - c_1 + c_0) + (6c_3 - 2c_2 + c_1) x + \sum_{n=2}^{\infty} \left[ (n+2)(n+1)c_{n+2} + \left(n^2 - n + 1\right)c_n - (n+1)c_{n+1} \right] x^n = 0
\]

Now
\(2c_2 - c_1 = c_0 = 0\), then \(c_2 = (c_1 - c_0)/2\)
\(6c_3 - 2c_2 + c_1 = 0\), then \(c_3 = (2c_2 - c_1)/6 = -c_0/6\)

\[
c_{n+2} = \frac{(n+1)c_{n+1} - (n^2 - n + 1)c_n}{(n+2)(n+1)}, \quad n \geq 2
\]
This last condition is called a recurrence formula, we can express each \( c_{n+2} \) in terms of a previous coefficient \( c_{n+1} \) and \( c_n \).

\[
\begin{align*}
\text{n = 2, } c_4 &= \frac{3c_3 - 3c_2}{4 \cdot 3} = \frac{c_3 - c_2}{4} = -\frac{1}{6}c_0 - \frac{1}{2}(c_1 - c_0) = \frac{2c_0 - 3c_1}{128} \\
\text{n = 3, } c_5 &= \frac{4c_4 - 7c_3}{5 \cdot 4} = \frac{4}{5 \cdot 4} \left( \frac{2c_0 - 3c_1}{24} \right) + \frac{7c_0}{6} = \frac{3c_0 - c_1}{40} \\
\text{n = 4, } c_6 &= \frac{36c_1 - 17c_0}{720}
\end{align*}
\]

Then, the general solution is:

\[
y(x) = c_0 + c_1x + \frac{c_1 - c_0}{2}x^2 - \frac{c_0}{6}x^3 + \frac{2c_0 - 3c_1}{24}x^4 + \frac{3c_0 - c_1}{40}x^5 + \frac{36c_1 - 17c_0}{720} + \ldots = \\
c_0 \left( 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 + \ldots \right) + c_1 \left( x + \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{40}x^5 + \ldots \right)
\]

5) Solve the non-homogeneous equation

\[y'' - 2x^2y' + 4xy = x^2 + 2x + 2\]

in powers of \( x \).

There are no singular points of the equation, so we want to find the series solution of the equation near zero. The series converges for all \( x \).

So \( y(x) = \sum_{n=0}^{\infty} c_n x^n \), \( y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1} \), \( y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \)

Substituting the series in the equation

\[
\begin{align*}
\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x^2 \sum_{n=1}^{\infty} nc_n x^{n-1} + 4x \sum_{n=0}^{\infty} c_n x^n - x^2 - 2x - 2 &= 0 \\
\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} 2nc_n x^{n+1} + \sum_{n=0}^{\infty} 4c_n x^{n+1} - x^2 - 2x - 2 &= 0 \\
\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=2}^{\infty} 2(n-1)c_{n-1}x^n + \sum_{n=1}^{\infty} 4c_{n-1}x^n - x^2 - 2x - 2 &= 0
\end{align*}
\]

We shift the index of summation in the first series by 2, replacing \( n \) with \( n + 2 \) and using the initial value \( n = 0 \). We shift the index of summation in the second series by -1, replacing \( n \) by \( n - 1 \) and using the initial value \( n = 2 \). We shift the index of summation in the third series by -1, replacing \( n \) by \( n - 1 \) and using the initial value \( n = 1 \).

Since we want to express everything in only one summation sign, we have to start the summation at \( n = 3 \) in every series (notice the term \( x^2 \) by itself),
\[ \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum_{n=2}^{\infty} 2(n-1)c_{n-1}x^n + \sum_{n=1}^{\infty} 4c_{n-1}x^n - x^2 - 2x - 2 = 0 \]

\[ 2c_2 + 6c_3x + \sum_{n=2}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum_{n=2}^{\infty} 2(n-1)c_{n-1}x^n + 4c_0x + \sum_{n=2}^{\infty} 4c_{n-1}x^n - x^2 - 2x - 2 = 0 \]

\[ (2c_2 - 2) + (6c_3 + 4c_0 - 2)x + (-1 + 12c_4 + 2c_1)x^2 + \sum_{n=3}^{\infty} [(n+2)(n+1)c_{n+2} - 2(n-1)c_{n-1} + 4c_{n-1}]x^n = 0 \]

Now

\[ 2c_2 - 2 = 0, \text{ then } c_2 = 1 \]
\[ 6c_3 + 4c_0 - 2 = 0, \text{ then } c_3 = 1/3 - 2/3 c_0 \]
\[ -1 + 12c_4 + 2c_1 = 0, \text{ then } c_4 = 1/12 - 1/6 c_1 \]
\[ (n + 2)(n + 1) c_{n+2} - 2(n - 3) c_{n-1} = 0, \text{ n} \geq 3 \]

\[ c_{n+2} = \frac{2(n-3)}{(n+1)(n+2)} c_{n-1}, \text{ n} \geq 3 \]

\[ n = 3, \text{ } c_5 = 0 \]
\[ n = 4, \text{ } c_6 = 2/30 \text{ } c_4 = 1/15(1/3 - 2/3 c_0) = 1/45 - 2/45c_0 \]
\[ n = 5, \text{ } c_7 = 4/42 \text{ } c_4 = 1/126 - 1/63 c_1 \]

\[ y(x) = c_0 + c_1x + x^2 + \left(\frac{2}{3} c_0\right)x^3 + \left(\frac{1}{12} - \frac{1}{6} c_1\right)x^4 + \left(\frac{1}{45} - \frac{2}{45} c_0\right)x^6 + \left(\frac{1}{126} - \frac{1}{63}\right)x^7 \ldots \]

\[ c_0 \left(1 - \frac{2}{3} x^3 - \frac{2}{45} x^6 + \ldots \right) + c_1 \left(x - \frac{1}{6} x^4 - \frac{1}{126} x^7 \ldots \right) + (x^2 + \frac{1}{3} x^3 + \frac{1}{12} x^4 + \frac{1}{45} x^6 + \frac{1}{126} x^7 \ldots) \]

6) Solve the non-homogeneous equation

\[ y'' - xy' + 2y = e^{-x} \]

in powers of x.

There are no singular points of the equation, so we want to find the series solution of the equation near zero. The series converges for all x.

So \( y(x) = \sum_{n=0}^{\infty} c_n x^n \), \( y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} \), \( y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \), \( e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} \)

Substituting the series in the equation

\[ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x \sum_{n=1}^{\infty} n c_n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 0 \]

\[ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} 2 c_n x^n - \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 0 \]

\[ \sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2} x^n - \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} 2 c_n x^n - \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!} = 0 \]

We shift the index of summation in the first series by 2, replacing n with n + 2 and using the initial value n = 0. Since we want to express everything in only one summation sign, we have to start the summation at n = 1 in every series,
\[
\sum_{n=0}^{\infty} (n+2)(n+1)c_{n+2}x^n - \sum_{n=1}^{\infty} nc_nx^n + \sum_{n=0}^{\infty} 2c_nx^n - \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} = 0
\]

\[
(2c_2 + 2c_0 - 1) + \sum \left[ (n+2)(n+1)c_{n+2} - (n-2)c_n - \frac{(-1)^n}{n!} \right] x^n = 0
\]

Now, \(2c_2 + 2c_0 - 1 = 0\), then \(c_2 = \frac{1}{2} - \frac{1}{2} c_0\)

\[(n+2)(n+1)c_{n+2} - (n-2)c_n - \frac{(-1)^n}{n!} = 0, \ n \geq 1\]

\[c_{n+2} = \frac{(n-2)c_n + \frac{(-1)^n}{n!}}{(n+2)(n+1)}, \ n \geq 1\]

\[n = 1, \ c_3 = \frac{-c_1 - 1}{3 \cdot 2} = \frac{-c_1}{6} - \frac{1}{6}\]

\[n = 2, \ c_4 = \frac{\frac{1}{2}}{4 \cdot 3} = \frac{1}{24}\]

\[n = 3, \ c_5 = \frac{c_3 - \frac{1}{6}}{5 \cdot 4} = \frac{c_3}{20} - \frac{1}{120} = \frac{1}{20} \left( \frac{-c_1}{6} - \frac{1}{6} \right) - \frac{1}{120} = - \frac{c_1}{120} - \frac{1}{60}\]

\[n = 4, \ c_6 = \frac{2c_4 + \frac{1}{24}}{6 \cdot 5} = \frac{c_4}{15} + \frac{1}{720} = \frac{1}{15 \cdot 24} + \frac{1}{720} = \frac{3}{720} = \frac{1}{240}\]

\[n = 5, \ c_7 = \frac{3c_5 - \frac{1}{120}}{7 \cdot 6} = \frac{c_5}{14} - \frac{1}{5040} = \frac{1}{14} \left( - \frac{c_1}{120} - \frac{1}{60} \right) - \frac{1}{5040} = - \frac{c_1}{1680} - \frac{7}{5040} = - \frac{c_1}{1680} - \frac{1}{720}\]

\[y(x) = c_0 + c_1x + \left( \frac{1}{2} - \frac{1}{2} c_0 \right) x^2 + \left( - \frac{c_1}{6} - \frac{1}{6} \right) x^3 + \frac{1}{24} x^4 + \left( - \frac{c_1}{120} - \frac{1}{60} \right) x^5 + \frac{1}{240} x^6 + \]

\[\left( - \frac{c_1}{1680} - \frac{1}{720} \right) x^7 + ....... =\]

\[c_0 \left( 1 - \frac{1}{2} x^2 \right) + c_1 \left( x - \frac{1}{6} x^3 - \frac{1}{120} x^5 - \frac{1}{1680} x^7 + .... \right) + \left( \frac{1}{2} x^2 - \frac{1}{6} x^3 + \frac{1}{24} x^4 - \frac{1}{60} x^5 - \frac{1}{720} x^7 + ... \right)\]

**Translated Series Solutions**

If we look for a solution of the equation \(P(x) \frac{d^2 y}{dx^2} + Q(x) \frac{dy}{dx} + R(x)y = 0\) of the form \(y = \sum_{n=0}^{\infty} c_n (x - a)^n\), we can make a change of variable \(x - a = t\), obtaining a new differential equation for \(y\) as a function of \(t\), and then look for solutions of this new
equation of the form $y(t) = \sum_{n=0}^{\infty} c_n t^n$. When we have finished the calculations, we replace $t$ by $x - a$.

**Example:**
Solve $2y''' + xy' + y = 0$ in powers of $x - 1$.
Notice that $x = 1$ is an ordinary point of the equation. We look for a solution in the form $y = \sum_{n=0}^{\infty} c_n (x - 1)^n$.

We can simplify the computation of the coefficients $c_n$ by translating the center of the expansion from $x = 1$ to $t = 0$.
This is accomplished by the substitution $x - 1 = t$ or $t = x + 1$, then $dt/dx = 1$ or $dt = dx$.
Since $y(x) = y(t + 1)$, using chain rule:

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot 1 = \frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left[ \frac{dy}{dt} \right] = \frac{d}{dt} \left[ \frac{dy}{dt} \right] \cdot \frac{dt}{dx} = \frac{d^2y}{dt^2}$$

So the equation transforms: $2 \frac{d^2y}{dt^2} + (t - 1) \frac{dy}{dt} + y = 0$.

We seek a solution in the form $y(t) = \sum_{n=0}^{\infty} c_n t^n$ where the coefficients $c_n$ are the same that appear in solution (2).

So $y(x) = \sum_{n=0}^{\infty} c_n t^n$, $y'(x) = \sum_{n=1}^{\infty} nc_n t^{n-1}$, $y''(x) = \sum_{n=2}^{\infty} n(n - 1)c_n t^{n-2}$
Substituting the series in the equation

$$2 \sum_{n=2}^{\infty} n(n - 1)c_n t^{n-2} + t \sum_{n=1}^{\infty} nc_n t^{n-1} - \sum_{n=1}^{\infty} nc_n t^{n-1} + \sum_{n=0}^{\infty} c_n t^n =$$

$$\sum_{n=2}^{\infty} 2n(n - 1)c_n t^{n-2} + \sum_{n=1}^{\infty} nc_n t^n - \sum_{n=1}^{\infty} nc_n t^{n-1} + \sum_{n=0}^{\infty} c_n t^n =$$

$$\sum_{n=0}^{\infty} 2(n + 2)(n + 1)c_{n+2} t^n + \sum_{n=1}^{\infty} nc_n t^n - \sum_{n=0}^{\infty} (n + 1)c_{n+1} t^n + \sum_{n=0}^{\infty} c_n t^n = 0$$

We shift the index of summation in the first series by 2, replacing $n$ with $n + 2$ and using the initial value $n = 0$. We shift the index of summation in the third series by 1, replacing $n$ by $n + 1$ and using the initial value $n = 0$.
Since we want to express everything in only one summation sign, we have to start the summation at $n = 1$ in every series,
\[
\sum_{n=0}^{\infty} 2(n+2)(n+1)c_{n+2}t^n + \sum_{n=1}^{\infty} nc_nt^n - \sum_{n=0}^{\infty} (n+1)c_{n+1}t^n + \sum_{n=0}^{\infty} c_nt^n =
\]

\[
4c_2 + \sum_{n=1}^{\infty} 2(n+2)(n+1)c_{n+2}t^n + \sum_{n=1}^{\infty} nc_nt^n - c_1 - \sum_{n=0}^{\infty} (n+1)c_{n+1}t^n + c_0 + \sum_{n=0}^{\infty} c_nt^n =
\]

\[
(4c_2 - c_1 + c_0) + \sum_{n=1}^{\infty} \left[2(n+2)(n+1)c_{n+2} + (n+1)c_n - (n+1)c_{n+1}\right]t^n = 0
\]

Now,
\[
4c_2 - c_1 + c_0 = 0, \text{ then } c_2 = \frac{1}{4}(c_1 - c_0)
\]

\[
2(n+2)(n+1)c_{n+2} + (n+1)(c_n - c_{n+1}) = 0, \ n \geq 1
\]

\[
c_{n+2} = -\frac{(n+1)(c_n - c_{n+1})}{2(n+2)(n+1)} = -\frac{c_n - c_{n+1}}{2(n+2)}, \ n \geq 1
\]

\[
n = 1, \ c_3 = -\frac{c_1 - c_2}{2 \cdot 3} = -\frac{c_1 - \frac{c_1 - c_0}{4}}{6} = -\frac{3c_1 + c_0}{24}
\]

\[
n = 2, \ c_4 = -\frac{c_2 - c_3}{2 \cdot 4} = -\frac{c_2 - \frac{c_1 - c_0}{4} + 3c_1 + c_0}{8} = -\frac{9c_1 - 5c_0}{192}
\]

\[
y(t) = c_0 + c_1t - \frac{c_1 - c_0}{4}t^2 - \frac{3c_1 + c_0}{24}t^3 - \frac{9c_1 - 5c_0}{192}t^4 - \ldots =
\]

\[
c_0 \left(1 + \frac{1}{4}t^2 - \frac{1}{24}t^3 + \frac{5}{192}t^4 + \ldots \right) + c_1 \left(t - \frac{1}{4}t^2 - \frac{1}{8}t^3 - \frac{9}{192}t^4 - \ldots \right)
\]

\[
y(x) = c_0 \left(1 + \frac{1}{4}(x-1)^2 - \frac{1}{24}(x-1)^3 + \frac{5}{192}(x-1)^4 + \ldots \right) +
\]

\[
c_1 \left((x-1) - \frac{1}{4}(x-1)^2 - \frac{1}{8}(x-1)^3 - \frac{9}{192}(x-1)^4 - \ldots \right)
\]