Special Integrating Factors

Given the O.D.E. \( M(x,y) \, dx + N(x,y) \, dy = 0 \), assume it is non-exact. Suppose that \( n(x,y) \) is an integrating factor of the equation, then

\[
n(x,y) \, M(x,y) \, dx + n(x,y) \, N(x,y) \, dy = 0
\]

is an exact equation.

Therefore,

\[
\frac{\partial}{\partial y} [n(x,y)M(x,y)] = \frac{\partial}{\partial x} [n(x,y)N(x,y)]
\]

or

\[
n(x,y) \frac{\partial M(x,y)}{\partial y} + \frac{\partial n(x,y)}{\partial y} M(x,y) = n(x,y) \frac{\partial N(x,y)}{\partial x} + \frac{\partial n(x,y)}{\partial x} N(x,y)
\]

or

\[
n(x,y) \left[ \frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} \right] = N(x,y) \frac{\partial n}{\partial x} - M(x,y) \frac{\partial n}{\partial y} \quad (1)
\]

\( n(x,y) \) is an unknown function that satisfies equation (1), but equation (1) is a partial differential equation. So, in order to find \( n(x,y) \) we have to solve a P.D.E. and we do not know how to do it.

Therefore, we have to impose some restriction on \( n(x,y) \).

Assume that \( n \) is function of only one variable, let’s say of the variable \( x \), then

\[
n(x) \quad \text{and} \quad \frac{\partial n}{\partial y} = 0, \quad \frac{\partial n}{\partial x} = \frac{dn}{dx}
\]

So, equation (1) reduces to

\[
n(x) \left[ \frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} \right] = N(x,y) \frac{dn}{dx}
\]

or

\[
\frac{1}{N(x,y)} \left[ \frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} \right] dx = \frac{dn}{n}
\]

If the left hand side of the above equation is only function of \( x \), then the equation is separable and \( n(x) = \exp \left( \int \frac{1}{N(x,y)} \left[ \frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} \right] dx \right) \).

Conclusion: The equation \( M(x,y) \, dx + N(x,y) \, dy = 0 \) has an integrating factor \( n(x) \) that depends only on \( x \) if the expression \( \frac{1}{N(x,y)} \left[ \frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} \right] \) depends only on \( x \).

Now, let’s assume the \( n \) depends only on the variable \( y \), then

\[
n(y) \quad \text{and} \quad \frac{\partial n}{\partial x} = 0, \quad \frac{\partial n}{\partial y} = \frac{dn}{dy}
\]

So, equation (1) reduces to
\[
\begin{align*}
\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} = -\frac{M(x,y)}{n(x)} \frac{dn}{dy}
\end{align*}
\]

or
\[
\frac{-1}{M(x,y)} \left[ \frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} \right] dy = \frac{dn}{n}
\]

or
\[
\frac{1}{M(x,y)} \left[ \frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y} \right] dy = \frac{dn}{n}
\]

If the left hand side of the above equation is only function of \( y \), then the equation is separable and
\[
n(y) = \exp \left( \int \frac{1}{M(x,y)} \left( \frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y} \right) dx \right).
\]

**Conclusion:** The equation \( M(x,y) \, dx + N(x,y) \, dy = 0 \) has an integrating factor \( n(y) \) that depends only on \( y \) if the expression \[
\frac{1}{M(x,y)} \left[ \frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y} \right]
\]
depends only on \( y \).

**Remark:** If neither of the above criteria is satisfied, then the equation has an integrating factor that depends on both variables \( x \) and \( y \), but it is impossible to determine at these level of the course.

**Examples:** Find the integrating factor

1) \( (4xy + 3y^2 - x) \, dx + x(x + 2y) \, dy = 0 \)
   
   \[
   M(x,y) = 4xy + 3y^2 - x \quad \text{and} \quad N(x,y) = x(x + 2y)
   \]
   
   \[
   \frac{\partial M(x,y)}{\partial y} = 4x + 6y \quad \text{and} \quad \frac{\partial N(x,y)}{\partial x} = 2x + 2y
   \]
   
   \[
   \frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} = 4x + 6y - 2x - 2y = 2x + 4y
   \]
   
   \[
   \frac{1}{N(x,y)} \left[ \frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} \right] = \frac{1}{x(x + 2y)} (2x + 4y) = \frac{2(x + 2y)}{x(x + 2y)} = \frac{2}{x}
   \]
   
   Since it depends on \( x \), only, there is an integrating factor \( n(x) \), given by
   
   \[
   n(x) = \exp \left( \int \frac{2 \, dx}{x} \right) = \exp \left( 2 \ln |x| \right) = x^2
   \]
   
   Multiply the original equation by \( n(x) \), we get the exact equation
   
   \[
   (4x^3y + 3x^2y^2 - x^3) \, dx + (x^4 + 2x^3y) \, dy = 0
   \]
   
   by grouping we get
   
   \[
   (4x^3y \, dx + x^4 \, dy) + (3x^2y^2 \, dx + 2x^3y \, dy) - x^3 \, dx = 0
   \]
   
   \[
   d(x^4y) + d(x^3y^2) - d(\frac{1}{4} x^4) = d(c)
   \]
   
   \[
   x^4y + x^3y^2 - \frac{1}{4} x^4 = c
   \]

2) \( y(x + y) \, dx + (x + 2y - 1) \, dy = 0 \)
   
   \[
   M(x,y) = y(x + y) \quad \text{and} \quad N(x,y) = x + 2y - 1
   \]
\[
\frac{\partial M(x,y)}{\partial y} = x + 2y \quad \text{and} \quad \frac{\partial N(x,y)}{\partial x} = 1
\]
\[
\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} = x + 2y - 1
\]
\[
\frac{1}{N(x,y)} \left[ \frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} \right] = \frac{1}{x + 2y - 1} (x + 2y - 1) = 1
\]

Since, the expression is constant, there is an integrating factor \( n(x) \)

\[
n(x) = e^{\int dx} = e^x
\]

Multiplying the original equation by \( n(x) \), we obtain the exact equation

\[
ye^x(x + y) \, dx + e^x(x + 2y - 1) \, dy = 0
\]

\[
F(x,y) = \int M(x,y) \, dx = \int \left( xy e^x + y^2 e^x \right) \, dx = y \left( xe^x - e^x \right) + y^2 e^x + B(y)
\]

\[
\frac{\partial F(x,y)}{\partial y} = N(x,y) = \frac{\partial}{\partial y} \left[ y \left( xe^x - e^x \right) + y^2 e^x + B(y) \right] = xe^x - e^x + 2ye^x + B'(y)
\]

then

\[
B'(y) = 0 \quad \text{and} \quad B(y) = c
\]

The solution is: \( xe^x - e^x + 2ye^x = k \)

3) \( y(x + y + 1) \, dx + x(x + 3y + 2) \, dy = 0 \)

\( M(x,y) = y(x + y + 1) \) \quad \text{and} \quad \( N(x,y) = x(x + 3y + 2) \)

\[
\frac{\partial M(x,y)}{\partial y} = x + 2y + 1 \quad \text{and} \quad \frac{\partial N(x,y)}{\partial x} = 2x + 3y + 2
\]

\[
\frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} = x + 2y + 1 - 2x - 3y - 2 = -(x + y + 1)
\]

\[
\frac{1}{N(x,y)} \left[ \frac{\partial M(x,y)}{\partial y} - \frac{\partial N(x,y)}{\partial x} \right] = \frac{-(x + y + 1)}{2x + 3y + 2}
\]

consider

\[
\frac{1}{M(x,y)} \left[ \frac{\partial N(x,y)}{\partial x} - \frac{\partial M(x,y)}{\partial y} \right] = \frac{1}{y(x + y + 1)} (x + y + 1) = \frac{1}{y}
\]

Since, it depends only on \( y \), there is an integrating factor \( n(y) \)

\[
n(y) = e^{\int dy} = e^{lny} = y
\]

Multiplying the original equation by \( n(y) \), we obtain the exact equation

\[
y^2(x + y + 1) \, dx + yx(x + 3y + 2) \, dy = 0
\]
\[
F(x,y) = \int M(x,y) \, dx = \int \left( xy^2 + y^3 + y^2 \right) \, dx = \frac{x^2}{2} y^2 + xy^3 + xy^2 + B(y)
\]
\[
\frac{\partial F(x,y)}{\partial y} = N(x,y) = \frac{\partial}{\partial y} \left[ \frac{x^2}{2} y^2 + xy^3 + xy^2 + B(y) \right] = x^2 y + 3xy^2 + 2xy + B'(y)
\]

Then \( B'(y) = 0 \) and therefore \( B(y) = c \)

The solution is: \( \frac{1}{2} x^2 y + xy^3 + xy^2 = k. \)