Special Transformation

There are certain equations that can be transformed into a more basic type using a suitable transformation. The equations have the form:

\[(a_1x + b_1y + c_1) \, dx + (a_2x + b_2y + c_2) \, dy = 0\]

where \(a_1, b_1, c_1, a_2, b_2, c_2\) are constants.

There are two different kinds of transformations according to relationships among the constants.

**Case 1:** \(\frac{a_2}{a_1} \neq \frac{b_2}{b_1}\)

Solve the system

\[a_1h + b_1k + c_1 = 0\]
\[a_2h + b_2k + c_2 = 0\]

because of the imposed condition the system has a unique solution \((h,k)\).

Then, the transformation:

\[x = X + h\]
\[y = Y + k\]

will change the original equation into a homogeneous equation in the variable \(X\) and \(Y\),

\[(a_1X + b_1Y) \, dX + (a_2X + b_2Y) \, dY = 0\]

**Case 2:** \(\frac{a_2}{a_1} = \frac{b_2}{b_1} = k\)

The transformation \(z = a_1x + b_1y\) changes the original equation into a separable equation in the variables \(z\) and \(x\).

**Examples:** Solve the equations

1) \((2x - 5x + 3) \, dx - (2x + 4y - 6) \, dy = 0\)

Since \(2/2 \neq 4/-5\), let’s solve the system

\[2h - 5k + 3 = 0\]
\[2h + 4k - 6 = 0\]

Subtract the second equation from the first one, to get

\[2h - 5h = -3\]
\[2h + 4k = 6\]
\[0 - 9k = -9\]

then \(k = 1\) and \(2h = -3 + 5\) or \(h = 1\).

So, the transformation:

\[x = X + 1, \quad dx = dX\]
\[y = Y + 1, \quad dy = dY\]

reduces the given equation to

\[(2X + 2 - 5Y - 5 + 3) \, dX - (2X + 2 + 4Y + 4 - 6) \, dY = 0\]
\[(2X - 5Y) \, dX - (2X + 4Y) \, dY = 0\]

which is homogeneous.

Using the transformation \(Y = VX\), and \(dY = VdX +XdV\),
We get the equation

\[
(2 - 5V) \, dX - (2 + 4V)(VdX + XdV) = 0 \\
(2 - 7V - 4V^2) \, dX - X(2 + 4V) \, dV = 0
\]

\[
dX - \frac{2 + 4V}{2 - 7V - 4V^2} \, dV = 0 \\
dX + \frac{4V + 2}{4V^2 + 7V - 2} \, dV = 0
\]

\[
\frac{4V + 2}{4V^2 + 7V - 2} = \frac{A}{V - 1} + \frac{B}{V + 2}
\]

\[
A = \frac{4}{3}, \quad B = \frac{2}{3}
\]

\[
dX + \frac{4}{3} \, dV + \frac{2}{3} \, dV = 0
\]

Integrating

\[
\ln|X| + \frac{1}{3} \ln|4V - 1| + \frac{2}{3} \ln|V + 2| = \ln|c|
\]

\[
X^3(4V - 1)(V + 2)^2 = k
\]

replacing \( V \) by \( \frac{Y}{X} \)

\[
X^3 \left( 4 \, \frac{Y}{X} - 1 \right) \left( \frac{Y}{X} + 2 \right)^2 = (4Y - X)(Y + 2)^2 = K
\]

replacing \( X \) by \( x - 1 \) and \( Y \) by \( y - 1 \),

\[
(4y - x - 3)(y + 2x - 3)^2 = K
\]

2) \((x + y) \, dx + (3x + 3y - 4) \, dy = 0\)

Since \(3/1 = 3/1\), we must take the transformation \( z = x + y \).

We use \( y = z - x \), \( dy = dz - dx \) to obtain

\[
z \, dx + (3z - 4) \, (dz - dx) = 0
\]

or

\[
z \, dx - 3z \, dx + 4 \, dx + (3z - 4) \, dz = 0
\]

or

\[
(4 - 2z) \, dx + (3z - 4) \, dz = 0
\]

and this is a separable equation,

Then

\[
2dx + \frac{3z - 4}{2 - z} \, dz = 2dx - \frac{3(z - 2) + 2}{z - 2} \, dz = 2dx - 3dz + \frac{2}{z - 2} \, dz = 0
\]

\[
2x - 3z + 2 \ln|z - 2| = c
\]

replacing \( z \) by \( x + y \),

\[
2x - 3(x + y) + 2\ln|x + y - 2| = c
\]

3) \((4x + 3x + 1) \, dx + (x + y + 1) \, dy = 0\)

Since \(1/4 \neq 1/3\), let’s solve the system
\[4h + 3k + 1 = 0\]
\[h + k + 1 = 0\]

Subtract the second equation multiplied by 4 from the first one, to get
\[4h + 3h = -1\]
\[4h + 4k = -4\]
\[0 - k = 3\]
then \(k = -3\) and \(h = 3 - 1\) or \(h = 2\).

So, the transformation:
\[x = X + 2, \quad dx = dX\]
\[y = Y - 3, \quad dy = dY\]

reduces the given equation to
\[(4X + 8 + 3Y - 9 + 1) dX + (X + 2 + Y - 3 + 1) dY = 0\]
\[= (4X + 3Y) dX + (X + Y) dY = 0\]
which is homogeneous.

Using the transformation \(Y = VX,\) and \(dY = VdX + XdV,\)
We get the equation
\[(4 + 3V) dX + (1 + V)(VdX + XdV) = 0\]
\[= (V^2 + 4V + 4) dX + X(1 + V) dV = 0\]

Integrating
\[\ln |X| + \ln |V + 2| + (V + 2)^{-1} = \ln |c|\]
\[\ln (|X||V + 2|c) = -(V + 2)^{-1}\]

replacing \(V\) by \(Y\)
\[\ln \left(\frac{Y}{X} + 2\right)^c = -\left(\frac{Y}{X} + 2\right)^{-1}\]
\[\ln (|Y + 2X|c)|X|^{-1} = -(Y + 2X)^{-1}\]

replacing \(X\) by \(x - 2\) and \(Y\) by \(y + 3,\)
\[\ln (|y + 3 + 2(x - 2)|c)|x - 2|^{-1} = -(y + 3 + 2(x - 2))^{-1}\]
\[\ln (|y + 2x - 1|c)|x - 2|^{-1} = -(y + 2x - 1)^{-1}\]