Variation of the Parameters

This is another method we can apply to find a particular solution of a non-homogeneous L.D.E..

We will develop the method in connection with a second order linear O.D.E. with variables coefficients,

\[ a_0(x)y'' + a_1(x)y' + a_2(x)y = b(x) \] (1)

Suppose that \( y_1(x) \) and \( y_2(x) \) are linearly independent solutions of the corresponding homogeneous equation,

\[ a_0(x)y'' + a_1(x)y' + a_2(x)y = 0 \]

then the complementary solution of equation (1) is:

\[ y_c(x) = c_1y_1(x) + c_2y_2(x) \]

where \( c_1 \) and \( c_2 \) are arbitrary constants.

Replace the arbitrary constants \( c_1 \) and \( c_2 \) in the complementary solution by two functions \( v_1(x) \) and \( v_2(x) \) to be determined such that

\[ y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \]

is a particular solution of equation (1).

This is the first condition new impose in order to determine \( v_1(x) \) and \( v_2(x) \). Since there are two functions, we still have a second condition to impose, provided that this second condition does not violate the first condition.

Let’s differentiate the function \( y_p \):

\[ y_p' = v_1(x)y_1'(x) + v_1'(x)y_1(x) + v_2(x)y_2'(x) + v_2'(x)y_2(x) \]

At this point we impose the second condition. We simplify \( y_p' \) by demanding

\[ v_1'(x)y_1'(x) + v_2'(x)y_2'(x) = 0 \]

So, \( y_p' \) becomes

\[ y_p' = v_1(x)y_1'(x) + v_2(x)y_2'(x) \]

And the second derivative is

\[ y_p'' = v_1(x)y_1''(x) + v_1'(x)y_1'(x) + v_2(x)y_2''(x) + v_2'(x)y_2'(x) \]

Substituting in equation (1)

\[ a_0(x)[v_1(x)y_1''(x) + v_1'(x)y_1'(x) + v_2(x)y_2''(x) + v_2'(x)y_2'(x)] + a_1(x)[v_1(x)y_1'(x) + v_2(x)y_2'(x)] + a_2(x)[v_1(x)y_1(x) + v_2(x)y_2(x)] = b(x) \]

or

\[ v_1(x)[a_0(x)y_1''(x) + a_1(x)y_1'(x) + a_2(x)y_1(x)] + v_2(x)[a_0(x)y_2''(x) + a_1(x)y_2'(x) + a_2(x)y_2(x)] + a_0(x)[v_1'(x)y_1'(x) + v_2'(x)y_2'(x)] = b(x) \]

Since \( y_1 \) and \( y_2 \) are solutions of the corresponding homogeneous equation, the first two terms on the left side of the equation are zero.

Then,

\[ a_0(x)[v_1'(x)y_1'(x) + v_2'(x)y_2'(x)] = b(x) \]

or

\[ v_1'(x)y_1'(x) + v_2'(x)y_2'(x) = b(x)/a_0(x) \]

So, the two imposed conditions have created a system of two equations that the derivatives of the two unknown functions \( v_1 \) and \( v_2 \) are satisfying.
\[
\begin{align*}
&v_1'(x)y_1(x) + v_2'(x)y_2(x) = 0 \\
&v_1'(x)y_1'(x) + v_2'(x)y_2'(x) = \frac{b(x)}{a_0(x)}
\end{align*}
\]

The determinant of the system is the Wronskian of \(y_1\) and \(y_2\)
\[
W(y_1,y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0
\]
since \(y_1\) and \(y_2\) are linearly independent. Then the system has a unique solution.

\[
v_1'(x) = \frac{-y_1'(x) + b(x)y_2(x)}{a_0(x)W[y_1,y_2]} = \frac{-b(x)y_2(x)}{a_0(x)W[y_1,y_2]}
\]

and

\[
v_2'(x) = \frac{y_1(x) - b(x)y_1(x)}{a_0(x)W[y_1,y_2]} = \frac{b(x)y_1(x)}{a_0(x)W[y_1,y_2]}
\]

Then

\[
v_1(x) = \int v_1'(x) \, dx = -\int \frac{b(x)y_2(x)}{a_0(x)W[y_1,y_2]} \, dx
\]

and

\[
v_2(x) = \int v_2'(x) \, dx = \int \frac{b(x)y_1(x)}{a_0(x)W[y_1,y_2]} \, dx
\]

**Examples:**

Solve the non-homogeneous L.O.D.E.,

1) \(y''' + y = \sec x \tan x\)

Corresponding homogeneous equation: \(y''' + y = 0\)

Characteristic equation: \(r^2 + 1 = 0\)

Zeros are: \(r_1 = i\) and \(r_2 = -i\)

Complementary solution: \(y_c(x) = c_1 \cos x + c_2 \sin x\).

Non-homogeneous term is: \(b(x) = \sec x \tan x\).

Assume \(y_p(x) = v_1(x) \cos x + v_2(x) \sin x\) is a particular solution of the equation, where the functions \(v_1\) and \(v_2\) will be determined at the end of the process.

Condition 1 is \(y_p(x)\) is a particular solution of the equation.

Let’s compute the first derivative of \(y_p(x)\):

\[
y_p'(x) = -v_1(x) \sin x + v_2(x) \cos x + v_1'(x) \cos x + v_2'(x) \sin x.
\]

Let’s impose condition 2:

\[
v_1'(x) \cos x + v_2'(x) \sin x = 0
\]

so \(y_p\) becomes

\[
y_p'(x) = -v_1(x) \sin x + v_2(x) \cos x
\]
Compute the second derivative of $y_p(x)$:

$$y_p''(x) = -v_1(x) \cos x - v_2(x) \sin x - v_1'(x) \sin x + v_2'(x) \cos x.$$  

Substitute into the equation:

$$-v_1(x) \cos x - v_2(x) \sin x - v_1'(x) \sin x + v_2'(x) \cos x = - v_1'(x) \sin x + v_2'(x) \cos x = \sec x \tan x.$$  

Thus, we obtain a system of two equations in the unknowns $v_1'$ and $v_2'$:

$$v_1'(x) \cos x + v_2'(x) \sin x = 0$$

$$- v_1'(x) \sin x + v_2'(x) \cos x = \sec x \tan x$$

Solving the system:

$$v_1'(x) = \begin{bmatrix} 0 & \sin x \\ \sec x \tan x & \cos x \\ \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = -\sin x \sec x \tan x \frac{1}{-\tan^2 x} = -\tan^2 x$$

$$v_2'(x) = \begin{bmatrix} \cos x & 0 \\ -\sin x & \sec x \tan x \\ \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} = \cos x \sec x \tan x \frac{1}{1} = \tan x$$

then,

$$v_1(x) = \int v_1'(x) \, dx = \int -\tan^2 x \, dx = -\int (\sec^2 x - 1) \, dx = -\tan x + x = x - \tan x$$

$$v_2(x) = \int v_2'(x) \, dx = \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\ln|\cos x| = \ln|\sec x|$$

Then, $y_p(x) = (x - \tan x) \cos x + \ln|\sec x| \sin x$

and the general solution is:

$$y(x) = y_c(x) + y_p(x) = c_1 \cos x + c_2 \sin x + (x - \tan x) \cos x + \ln|\sec x| \sin x.$$  

2) Consider the equation with variable coefficients

$$(x^2 + 1) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 6(x^2 + 1)^2$$

The corresponding homogeneous equation has a solution,

$$y_c(x) = c_1 x + c_2 (x^2 - 1)$$

Assume $y_p(x) = v_1(x) x + v_2(x) (x^2 - 1)$ is a particular solution of the equation, where the functions $v_1$ and $v_2$ will be determined at the end of the process.

Condition 1 is $y_p(x)$ is a particular solution of the equation.

Let’s compute the first derivative of $y_p(x)$:

$$y_p'(x) = v_1(x) + v_2(x) 2x + v_1'(x) x + v_2'(x) (x^2 - 1).$$

Let’s impose condition 2:
\[ v_1'(x) x + v_2'(x) (x^2 - 1) = 0 \]

so \( y_p' \) becomes
\[ y_p'(x) = v_1(x) + v_2(x) \cdot 2x \]

Compute the second derivative of \( y_p(x) \):
\[ y_p''(x) = v_1'(x) + v_2'(x) \cdot 2x + 2v_2(x) \]

Substitute into the equation:
\[
(x^2 + 1) (v_1'(x) + v_2'(x) \cdot 2x + 2v_2(x)) - 2x(v_1(x) + v_2(x) \cdot 2x) + 2(v_1(x) x + v_2(x) (x^2 - 1)) =
(x^2 + 1) (v_1'(x) + v_2'(x) \cdot 2x) + 2v_2(x)x^2 + 2v_2(x) - 2xv_1(x) - 4x^2v_2(x) + 2xv_1(x) + 2x^2v_2(x) - 2v_2(x) = 6(x^2 + 1)^2.
\]

or
\[
(x^2 + 1) (v_1'(x) + v_2'(x) \cdot 2x) = 6(x^2 + 1)^2
\]

and dividing by \( x^2 + 1 \)
\[ v_1'(x) + 2x v_2'(x) = 6(x^2 + 1) \]

Thus, we obtain a system of two equations in the unknowns \( v_1' \) and \( v_2' \):
\[
\begin{align*}
v_1'(x) x + v_2'(x) (x^2 - 1) &= 0 \\
v_1'(x) + 2x v_2'(x) &= 6(x^2 + 1)
\end{align*}
\]

Solving the system:
\[
\begin{bmatrix}
0 & x^2 - 1 \\
6(x^2 + 1) & 2x
\end{bmatrix}
\begin{bmatrix}
v_1'(x) \\
v_2'(x)
\end{bmatrix}
= 
\begin{bmatrix}
-6(x^2 + 1)(x^2 - 1) \\
-6(x^2 + 1)(x^2 - 1)
\end{bmatrix}
= 
\begin{bmatrix}
-6(x^2 - 1) \\
-6(x^2 - 1)
\end{bmatrix}
\]

\[
\begin{bmatrix}
x & 0 \\
1 & 6(x^2 + 1)
\end{bmatrix}
\begin{bmatrix}
v_1'(x) \\
v_2'(x)
\end{bmatrix}
= 
\begin{bmatrix}
x(x^2 + 1) \\
2x^2 - x^2 + 1
\end{bmatrix}
= 
\begin{bmatrix}
x^2 + 1 \\
x^2 + 1
\end{bmatrix}
= 6x
\]

then
\[ v_1(x) = -2x^3 + 6x \] and \( v_2(x) = 3x^2 \)

Then, \( y_p(x) = (-2x^3 + 6x) x + 3x^2 (x^2 - 1) = x^4 + 3x^2 \)

and the general solution is:
\[ y(x) = y_c(x) + y_p(x) = c_1 x + c_2 (x^2 - 1) + x^4 + 3x^2. \]

3) \( y''' - 3y' + 2y = (1 + e^x)^{-1} \)

Corresponding homogeneous equation: \( y''' - 3y' + 2y = 0 \)

Characteristic equation: \( r^3 - 3r^2 + 2 = (r - 1)(r - 2) \)

Zeros are: \( r_1 = 1, \) and \( r_2 = 2 \)

Complementary solution: \( y_c(x) = c_1 e^x + c_2 e^{2x}. \)

Non-homogeneous term is: \( b(x) = (1 + e^x)^{-1}. \)

Assume \( y_p(x) = v_1(x) e^x + v_2(x) e^{2x} \) is a particular solution of the equation, where the functions \( v_1 \) and \( v_2 \) will be determined at the end of the process.

Condition 1 is \( y_p(x) \) is a particular solution of the equation.
Let’s compute the first derivative of \( y_p(x) \):

\[
y_p'(x) = v_1(x) e^x + 2v_2(x) e^{2x} + v_1'(x) e^x + v_2'(x) e^{2x}.
\]

Let’s impose condition 2:

\[
v_1'(x) e^x + v_2'(x) e^{2x} = 0
\]

so \( y_p' \) becomes

\[
y_p'(x) = v_1(x) e^x + v_2(x) e^{2x}
\]

Compute the second derivative of \( y_p(x) \):

\[
y_p''(x) = v_1(x) e^x + 4v_2(x) e^{2x} + v_1'(x) e^x + v_2'(x) e^{2x}.
\]

Substitute into the equation:

\[
v_1(x) e^x + 4v_2(x) e^{2x} + v_1'(x) e^x + v_2'(x) e^{2x} + 3v_1(x) e^x + 3v_2(x) e^{2x} + 2v_1(x) e^x + 2v_2(x) e^{2x}
\]

\[
= v_1'(x) e^x + 2v_2'(x) e^{2x} = (1 + e^x)^{-1}.
\]

Thus, we obtain a system of two equations in the unknowns \( v_1' \) and \( v_2' \):

\[
\begin{align*}
v_1'(x) e^x + v_2'(x) e^{2x} &= 0 \\
v_1'(x) e^x + 2v_2'(x) e^{2x} &= (1 + e^x)^{-1}
\end{align*}
\]

Solving the system:

\[
v_1'(x) = \begin{vmatrix} 0 & e^{2x} \\ (1 + e^{-x})^{-1} & e^{2x} \\ e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = \frac{-e^{2x}(1 + e^{-x})^{-1}}{e^x} = \frac{e^{-x}}{1 + e^{-x}}
\]

\[
v_2'(x) = \begin{vmatrix} e^x & 0 \\ e^{2x} (1 + e^{-x})^{-1} & e^{2x} \\ e^x & e^{2x} \\ e^{2x} & 2e^{2x} \end{vmatrix} = \frac{e^x(1 + e^{-x})^{-1}}{e^x} = \frac{e^{-2x}}{1 + e^{-x}}
\]

then,

\[
v_1(x) = \int v_1'(x) dx = \int \frac{-e^{-x}}{1 + e^{-x}} dx = \ln|1 + e^{-x}|
\]

\[
v_2(x) = \int v_2'(x) dx = \int \frac{e^{-2x}}{1 + e^{-x}} dx = \int \frac{e^{-x}}{1 + e^{-x}} e^{-x} dx = -\int \frac{u}{u + 1} du = -\int \left[ u + 1 - \frac{1}{u + 1} \right] du = \ln|1 + e^{-x}| - e^{-x}, \quad \text{if } u = 1 + e^{-x}, \ du = -e^{-x}
\]

Then, \( y_p(x) = e^x \ln|1 + e^x| - e^x + e^{2x} \ln|1 + e^x| \)

and the general solution is:

\[
y(x) = y_c(x) + y_p(x) = c_1 e^x + c_2 e^{2x} + e^x \ln|1 + e^x| - e^x + e^{2x} \ln|1 + e^x|
\]

**Remark:** This method can be extended to higher-order linear equations. New conditions must be imposed.
4) \( y''' + y' = \csc x \)
Corresponding homogeneous equation: \( y''' + y' = 0 \)
Characteristic equation: \( r^3 + r = r(r^2 + 1) = 0 \)
Zeros are: \( r_1 = 0, \ r_2 = i, \ r_3 = -i. \)
Complementary solution: \( y_c(x) = c_1 + c_2 \cos x + c_3 \sin x. \)
Non-homogeneous term is: \( b(x) = \csc x. \)
Assume \( y_p(x) = v_1(x) + v_2(x) \cos x + v_3(x) \sin x \) is a particular solution of the equation, where the functions \( v_1, v_2, \) and \( v_3 \) will be determined at the end of the process.
Condition 1 is \( y_p(x) \) is a particular solution of the equation.
Let’s compute the first derivative of \( y_p(x): \)
\[ y_p'(x) = -v_2(x) \sin x + v_3(x) \cos x + v_1'(x) + v_2'(x) \cos x + v_3'(x) \sin x. \]
Let’s impose condition 2:
\[ v_1'(x) + v_2'(x) \cos x + v_3'(x) \sin x = 0 \]
so \( y_p' \) becomes
\[ y_p'(x) = -v_2(x) \sin x + v_3(x) \cos x \]
Compute the second derivative of \( y_p(x): \)
\[ y_p''(x) = -v_2(x) \cos x - v_3(x) \sin x - v_2'(x) \sin x + v_3'(x) \cos x. \]
Let’s impose condition 3:
\[ -v_2'(x) \sin x + v_3'(x) \cos x = 0 \]
so \( y_p'' \) becomes
\[ y_p''(x) = -v_2(x) \cos x - v_3(x) \sin x \]
Compute the third derivative of \( y_p(x) \)
\[ y_p'''(x) = -v_2'(x) \cos x - v_3'(x) \sin x + v_2(x) \sin x - v_3(x) \cos x \]
Substitute into the equation:
\[ -v_2'(x) \cos x - v_3'(x) \sin x + v_2(x) \sin x - v_3(x) \cos x + v_2'(x) \sin x - v_3'(x) \cos x = -v_2'(x) \cos x - v_3'(x) \sin x = \csc x \]
Thus, we obtain a system of three equations in the unknowns \( v_1', v_2' \) and \( v_3': \)
\[ v_1'(x) + v_2'(x) \cos x + v_3'(x) \sin x = 0 \]
\[ -v_2'(x) \sin x + v_3'(x) \cos x = 0 \]
\[ -v_2'(x) \cos x - v_3'(x) \sin x = \csc x \]
Solving the system:
\[
\begin{bmatrix}
0 & \cos x & \sin x \\
0 & -\sin x & \cos x \\
\csc x & -\cos x & -\sin x
\end{bmatrix}
\begin{bmatrix}
v_1'(x) \\
v_2'(x) \\
v_3'(x)
\end{bmatrix}
= \csc x
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\]
\[ v_2'(x) = \begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \csc x & -\sin x \end{vmatrix} = \frac{-\cot x}{1} \]

\[ v_3'(x) = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \csc x \end{vmatrix} = \frac{-1}{1} = -1 \]

then,

\[ v_1(x) = \int \csc x \, dx = \ln (\csc x - \cot x) \]

\[ v_2(x) = \int -\cot x \, dx = -\ln |\sin x| = \ln |\csc x| \]

\[ v_3(x) = \int -dx = -x \]

So,

\[ y_p(x) = \ln(\csc x - \cot x) + \ln |\csc x| \cos x - x \sin x. \]

The general solution is:

\[ y(x) = y_c(x) = y_p(x) = c_1 + c_2 \cos x + c_3 \sin x + \ln(\csc x - \cot x) + \ln |\csc x| \cos x - x \sin x \]