Poisson and GaussianStatistics

Senior Lab

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1 Goal

In this experiment you will explore the statistics of random events. The random events used in this study will be generated by charged particles from a radioactive source detected by a scintillation detector connected to a photomultiplier tube (PMT).

2 Introduction

In the decay of some radioactive elements, the emission of charged particles occurs in a random way: the emission of one charged particle is completely independent of the emission of another. ^{90}Sr is one such element. ^{90}Sr decays by the emission of 546 keV electrons (beta decay) [1]. In other sources, ^{235}U, for example, there is a correlated decay chain in which the ^{235}U nucleus decays to ^{231}Th by alpha emission, which, in turn, beta decays into ^{231}Pa. In this case, the beta decays are correlated with the alpha decays of the parent nucleus.

A continuous random process is said to have a "steady-state decay rate" with mean rate μ if

$$\lim_{t\to\infty}\left(\frac{N}{t}\right) = \mu \tag{1}$$

where N is the number of events accumulated in time t. One can determine if a process is steady by taking repeated measurements of the number of counts in a given time interval and seeing if there is a trend in the successive values of the rate $R = N/t$.

Now the rate is certain to have statistical fluctuations and repeated measurements of R will rarely get exactly the same value. If the rate is a steady-state rate, the rates will be distributed according to Poisson statistics. The Poisson probability distribution is given by [2]

$$P(x;\mu) = \frac{\mu^x e^{-\mu}}{x!} \tag{2}$$

where $P(x;\mu)$ is the probability of recording x for a mean μ. In our case, μ is the mean rate and x is the rate. The standard deviation of the Poisson distribution is given by $\sigma = \sqrt{\mu}$. Note that when generating a Poisson distribution, x must be an integer.

The standard deviation of any distribution is given by

$$\sigma^2 = \langle x^2\rangle - \mu^2 \tag{3}$$

where $\langle x^2 \rangle$ is the average value of x^2. The standard deviation tells us something about how data are distributed about the mean. That is, a large standard deviation indicates many data points far from the mean and that the distribution is "broad". On the other hand, a small standard deviation indicates most of the data point points fall close to the mean and we would say the distribution is "narrow".

In the limit of large μ, the Poisson distribution may be approximated by a Gaussian or Normal distribution:

$$P(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]. \tag{4}$$

In addition, a given measurement of the rate $R_i = N_i/t_i$ will have an uncertainty that is related to the number of counts. For a rate governed by Poisson statistics, the uncertainty in the number of counts, N_i is given by

$$\sigma_{N_i} = \sqrt{N_i} \tag{5}$$

and the subsequent uncertainty in the rate is given by

$$\sigma_{R_i} = \frac{\sqrt{N_i}}{t_i}. \tag{6}$$

(There is, of course, some *systematic* uncertainty associated with t_i that will also contribute to the uncertainty in R_i, but for now, we will ignore this.) In general, the uncertainty in any value of a function $f(x)$ is given by $\sigma_f = f(\sigma_x)$. On the average for a Gaussian distribution, approximately 68% of a series of individual measurements of the rate will fall within one standard deviation of the true mean and 95% will fall within two standard deviations. Conversely, a single measurement has a 68% probability of falling within σ_{R_i} of the true mean.

You should be careful not to confuse uncertainties with standard deviations. This confusion is common because people frequently use the same symbols, σ_x for both. I used σ above in equations 5 and 6 for uncertainty. Some people prefer to avoid the confusion altogether and use the Δx or δx for the uncertainty.

You will do three sets of 100 measurements of the rate for beta decay of ^{90}Sr (plus cosmic-ray background) and determine the mean rate and see if the rate follows Poisson or Gaussian statistics.

3 Experiment

You will use the following equipment:

- Scintillation counter with PMT

- High voltage supply
- Discriminator
- Counter/timer
- Oscilloscope
- ^{90}Sr source

We will discuss the setup of the equipment during class. The ^{90}Sr is relatively safe–provided you don't eat it. However, care should always be taken when handling radioactive sources. The source will be checked out to you at the start of class and must be returned to me at the end of class.

Do the following steps to take your measurement:

- Set the counter/timer to count for 1 second and adjust your set up so that you get a rate in the neighborhood of 1 Hz. Note that you may end up having the source very far away so that all you are really detecting are cosmic rays. That's fine since cosmic-ray rates are also uncorrelated and will follow Poisson statistics. Make 100 measurements for this set up.
- Adjust the distance between the source and scintillator so that you get a rate of approximately 10 Hz and make 100, 1-second measurements.
- Repeat your measurements for a rate of approximately 100 Hz.

Obviously, this will get a bit repetitive but shouldn't take too long.

4 Analysis

You will need to use some computer program such as an Excel spreadsheet to tabulate your results and perhaps something like Sigmaplot to plot and fit your results. I can also provide Linux-based FORTRAN routines and plotting programs if you prefer.

- Tabulate your data and calculate the mean and standard deviation for each data set.
- Make a histogram of "number of trials" (y-axis) versus "count rate" (x-axis).

- "Fit" theoretical distributions (Eq. 2 and Eq. 4) to your data and plot it on the same graph. Sigmaplot has a fitting function or you can use the fitting program found at `http://statpages.org/nonlin.html`. In principle, you should be able to fit a Poisson distributions by hand. Since you can calculate the mean, you can generate the probability distribution and all that is left is to "normalize" the data by multiplying by an appropriate factor so that the area under the curve is the same as the area under data. Note that generating the Poisson distribution for large x is a bit tricky because of the $x!$ in the denominator and the online fitter generally fails. So you will probably have to do the 100 Hz fit by hand using the recursion relation:

  $$P(x;\mu) = \frac{\mu}{x} P(x-1;\mu) \tag{7}$$

  This means that you have to find $P(x;\mu)$ at some small value of x and generate the values at larger values of x. This is pretty straight forward using a spreadsheet. Here is the step-by-step procedure:

  1. Find the mean of your data by $\mu = \sum(x_i N_i)/\sum N_i$ where N_i is the number of occurrences of a given number of counts, x_i. This can easily be done in a spreadsheet.
  2. Find $P(x=0;\mu) = e^{-\mu}$.
  3. Use this value to find $P(x=1;\mu) = \mu * P(x=0;\mu)/1$ in the next row of the spreadsheet.
  4. Repeat step 3 for all of your data. If you sum up the probabilities, you should get something pretty darn close to 1.
  5. So now you want the amplitude for each value so you need to put in a normalization factor. A good first guess is to try the total number of observations you did. You can calculate a χ^2 using your data and the uncertainties (remember, the uncertainty on 0 occurrences is 1). Then try adjusting the normalization factor by hand to minimize χ^2.

- Is your data representative of a Poisson distribution or is it better represented by a Gaussian distribution (Eq. 4)?

- In your report, be sure to compare the data to the theoretical expectation. Discuss any "out-lier" bins. Are the number of these guys reasonable? The use of χ^2/ν also tells how well a given theoretical distribution reproduces a data distribution. Please see the supplemental handout regarding χ^2/ν.

References

[1] See, for example, http://atom.kaeri.re.kr/ and http://www.nuclides.net/applets/radioactive_decay.htm.

[2] Philip R. Bevington and D. Keith Robinson, *Data Reduction and Error Analysis for the Physical Sciences, 2nd Ed.*, pp 23-28, (McGraw-Hill, Inc, 1992).