Capital Asset Pricing Model

Joel Barber

Department of Finance
Florida International University
Miami, FL 33199
Capital Asset Pricing Model

• Mean-variance efficient risky portfolio

  – For each asset \( j = 1, 2, ..., N \)
    \[
    \frac{E[R_j] - R_F}{\text{cov}(R_j, R_p)} = \lambda
    \]

  – Remember \( \text{cov}(R_j, R_p) \) is the risk asset \( j \) contributes to portfolio \( P \).

  – \( \lambda \) is a constant:
    \[
    \lambda = \frac{E[R_p] - R_F}{\sigma_p^2}
    \]

  – In a efficient portfolio each asset contributes the same excess return per unit of risk.

• Market portfolio \( M \) is a value weighted portfolio of all risky assets
Market weight in asset $j$:

$$w_j \equiv \frac{\text{market value of asset } j}{\text{market value of all assets}}$$

Example. Suppose GM market capitalization is $100$ billion dollars and market capitalization of all assets is $1000$ billion. Then GMs market weight equals .1.

If you wanted to invest in market portfolio, you would choose a portfolio of all assets with market weights.

For example, you would invest 10% of your wealth in GM.

Expected market return:

$$E[R_m] = \sum_{j=1}^{N} w_jE[R_j]$$
- Variance of market return:

\[
\sigma^2_m = \sum_{j=1}^{N} \sum_{k=1}^{N} w_j w_k \sigma_{jk}
\]

- Risk asset \( j \) contributes to market portfolio:

\[
cov(R_j, R_m)
\]

- Key result of the CAPM: the market portfolio is mean-variance efficient.

- Therefore

\[
\frac{E[R_j] - R_F}{cov(R_j, R_m)} = \lambda
\]

for each asset.

- Market price of risk

\[
\lambda = \frac{E[R_m] - R_F}{\sigma^2_m}
\]  

(1)
* Excess return on the market per unit of total risk

* Reflects level of risk aversion in economy

* Equity premium puzzle

- Solve for expected return:

\[ E[R_j] = R_F + \lambda \text{cov}(R_j, R_m) \quad (2) \]

- This gives us risk-return relationship.

- Investors are compensated for the time value of money \( R_F \) plus a risk premium, which equals a constant times the risk a given asset contributes to total market risk

- If we substitute (1) for \( \lambda \) in equation (2), we obtain the security market line (SML):

\[ E[R_j] = R_F + \frac{\text{cov}(R_j, R_m)}{\sigma_m^2}(E[R_m] - R_F) \]
- Beta:

\[
\beta_j = \frac{\text{cov}(R_j, R_m)}{\sigma_m^2}
\]

- Beta equals the ratio of the risk asset \( j \) contributes to the market portfolio to the total market risk.

- It is often estimated by regressing asset return on the market return

- Market model:

\[
R_{jt} = \alpha_j + \beta_j R_{mt}
\]

- Capital market line (CML) is the efficient frontier with risk-free asset

\[
E[R_E] = R_F + \left( \frac{E[R_m] - R_F}{\sigma_m} \right) \sigma_E
\]

- All investors choose the same portfolio of risky assets and adjust for individual risk preferences by borrowing or lending at the risk-free rate.
- Asset allocation problem is reduced to determining fraction of wealth allocated to the market portfolio.

- Important point
  - security market line holds only if market portfolio is efficient
  - the capital market line holds for any risky portfolio
  - however, the capital market line is the efficient frontier only if the market portfolio is efficient, which requires that the SML hold for each asset.

- Question: if the market portfolio is efficient, can an investor outperform the market portfolio?

- What do we mean by outperform?
Suppose actively managed portfolio has expected return $E[R_a]$ and risk $\sigma_a$.

Let’s compare actively managed to passive strategy with same risk.

The passive strategy consists of a combination of the market portfolio and risk free rate with risk $\sigma_a$.

* Invest

$$X = \frac{\sigma_a}{\sigma_m}$$

in the market portfolio and

$$(1 - X)$$

in the risk-free rate.

* We expected return on passive strategy is given by

$$R_F + \left( \frac{E[R_m] - R_F}{\sigma_m} \right) \sigma_a$$
* Define alpha as difference between the expected return on the active and passive strategy:

\[ \alpha = E[R_a] - R_F - \left( \frac{E[R_m] - R_F}{\sigma_m} \right) \sigma_a \]

- Now we can attempt to answer the question: if the market portfolio is efficient, can an investor outperform the market portfolio?

- Suppose an investor always chooses (in advance) the one asset that will have the highest return.

- Clearly, the investor will outperform the passive strategy. Her alpha will be quite large.

- But we might think that her superior performance was result of luck

- On the other hand, her superior performance could have been the result of better information.
- We need to separate superior information from luck.

- In determining the efficient frontier, the information set consists of the covariance matrix and the vector of expected returns.

- If the market portfolio is efficient, the average alpha of investors without superior information is zero.

- Because of good luck some investors will have positive alphas and because of bad luck some will have negative alphas. However, if the market is efficient the average alpha will be zero.

- An investor who follows the passive strategy will always have a zero alpha.
• Theory

  – Assumptions

    * investors choose portfolios on the basis of mean and variance
    * perfect markets: no taxes, no transaction costs, assets are perfectly divisible
    * homogenous information: all investors have the same information consisting of covariance matrix and vector of expected returns
    * homogenous time horizon
    * risk-free borrowing and lending
    * all assets trade
    * equilibrium

  – The first six assumptions imply that all investors choose the same portfolio of risky as-
sets and adjust for individual risk preferences by borrowing or lending at the risk-free rate

- Equilibrium requires that if all investors hold the same portfolio, that each investor holds the market portfolio.

- For example, if GM is 10% of the total market capitalization, then each investor must have invested 10% of his or her wealth in GM.

- Why? Suppose each investor chooses, as result of mean variance optimization, to hold 11% of wealth in GM. Now if we sum up everyone’s wealth it must equal the sum of capitalizations of each asset. We can determine total wealth in the economy by either adding up investor’s wealth or by adding up market capitalizations. So GM’s market capitalization must be 11%.

- Think of the market portfolio as a round layer cake.
– Each layer represents an asset and the volume of each layer represents the market capitalization.

– An individual investor’s portfolio is a piece of the cake.

– For example, if an investor puts 100% in GM, his piece of cake will come from the GM layer.

– If all investors hold the same portfolio, then each investor’s portfolio is a slice of the cake.

– The size of the slice is determined by investor’s wealth.

• Violation of assumptions