MODELING ECONOMIC TIME SERIES
WITH STABLE SHOCKS

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

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The assumption of normality is commonly motivated in econometrics on the basis of the central limit theorem and the fact that much of the data is aggregated across time and across agents. However, most economic time series indicate non-normality in the form of either occasional big shocks or marked changes in the behavior of the series over different sub-periods (regime shifts). Gaussian processes have tails that are too thin to accommodate big shocks and they are too sluggish to learn about regime changes. The inability to effectively account for either of these features can be costly.

In this thesis infinite variance stable shocks which retain the central limit attributes as well as the advantages of continuity are used to model univariately the U.S. inflation rate, the value-weighted CRSP real stock returns, and the quarterly U.S. real GNP. We examine in detail the relationship between the level of inflation and its uncertainty, persistence in stock returns, and non-linearities and long memory in GNP.

Results indicate that the assumption of normality can be rejected for all three series. We find that inferences on the relationship between the level of inflation and its uncertainty are qualitatively and quantitatively sensitive to distributional assumptions. Stock returns fail to reveal a statistically significant mean-reverting component, after accounting for leptokurtosis, seasonality, and volatility persistence. Real GNP is characterized by a non-linear propagation mechanism, with little evidence of long memory. We also find that, in comparison to Gaussian models, stable models account for outliers and level shifts better, give more realistic assessment of uncertainty associated with such episodes, and provide tighter estimates of model parameters.
Dedicated to my parents
in the hope that this will convince them that
it is, after all, o.k. to be a scholar!!!
ACKNOWLEDGMENTS

There are several individuals to whom I owe a great deal for their involvement in my successful completion of the Ph.D.

My deepest gratitude goes to Hu McCulloch for his many ideas without which this dissertation would have taken a very different outlook, for introducing me to stable distributions, and for endless hours of discussion which helped me tremendously in working through my maze of confusion on several topics. I am greatly indebted to G.S. Maddala for being my advisor at a very crucial stage of my Ph.D. program, for graciously providing me with the use of several of his resources, including computational hardware and software, and for helping me define a schedule for timely completion of my Ph.D. Many thanks go to Mario Crucini for getting me started early on my research and for teaching me most of the ropes in the early stages, and to Pok-sang Lam for clarifying several technical details during most stages of my work.

I would also like to thank the audiences at the Macroeconomics Seminars, the Econometrics Seminars and the Macro Lunch Workshops at the OSU Department of Economics, and at the UCSB Statistics Department Symposium on Stable Processes during December 1995 for helpful suggestions and comments on various parts of my dissertation.

I would like to acknowledge partial financial support from the Charles Dice Fellowship for dissertation research from the OSU Department of Economics.
Special thanks to Ravikanth Pappu at the MIT Media Lab, without whose acquaintance I would have given up on academics a long time ago, and the many other people who make all this seem worthwhile.

Lastly, thanks to Jo Ducey for helping me handle most of the pesky administrative details during the course of this long saga.
VITA

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CHAPTER 1

INTRODUCTION

The assumption of normality is commonly motivated in econometrics on the basis of the central limit theorem and the fact that much of the economic data is aggregated across time and across agents. It is well-known, however, that the normal distribution does not accurately reflect the empirical distribution of most economic time series, particularly with respect to the tail behavior of the series. Two features that account for the apparent non-normality in economic series are noteworthy. One is the presence of big shocks or outlying observations. Another is abrupt regime changes over different sub-periods.

Blanchard and Watson (1986) argue that evidence of excess kurtosis found in the residuals of a structural VAR that includes aggregate prices, output, money and a fiscal policy variable points to large infrequent shocks in the series. Balke and Fomby (1994) examine fifteen postwar U.S. macroeconomic time series for outliers using univariate autoregressive models and find significant excess kurtosis in all but one of the series (the money base), indicating the presence of outliers. They find that outliers account for more than 20 percent of the volatility in many of their series, with the percentage exceeding 50
for the GNP deflator, the CPI, nominal compensation and M2. A cursory look at inflation and nominal interest rates indicates marked changes in the behavior of the series over different sub-periods, indicating changes in regime governing the time series.

While outliers can be accounted for by assuming alternative fat-tailed distributions such as the $t$-distribution, and regime changes are often modeled in the literature using mixtures of normals or Markov regime switching models, these statistical distributions cannot arise from aggregates of random variables. The Generalized Central Limit Theorem delivers the result that the sum of iid random variables must have a limiting distribution from the stable family if such a limiting distribution exists (Zolotarev (1986, Ch.1)). Thus, the very attribute that makes the normality assumption so appealing in economics motivates the case for considering more general stable shocks (which include the normal as a special case) in modeling economic series.

The non-Gaussian members of the stable family all have thicker tails than the normal density. In fact for all but the normal distribution, the second moment does not exist. Stable models have been estimated for stock returns by Fama (1965), Buckle (1995), and Mantegna and Stanley (1995), for interest rate movements by McCulloch (1985), and for bond returns by Oh (1994). The present study demonstrates the feasibility of modeling economic time series with symmetric stable shocks using data on three important time series, namely, the CPI-based U.S inflation rate, the value-weighted CRSP stock returns, and the U.S. real GNP.

The use of stable distributions in empirical work so far has been somewhat cumbersome due to the lack of a fast numerical routine for computing the stable cumulative distribution and probability density functions. The most widely used methods were to evaluate Zolotarev’s (1986, p.74,78) proper integral representations or to invert the Fourier transform of the characteristic function. A recent fast numerical approximation to the symmetric stable distribution and density due to McCulloch (1994b) opens the door for routine empirical analysis with symmetric stable disturbances.

This dissertation is organized into three main chapters. In Chapter 2 we univariately model the U.S. inflation rate with stable disturbances. We first set up a
homoskedastic state-space model for observed inflation, with the expected inflation rate as the state variable. Although a Gaussian state-space can be estimated using the Kalman recursions, the non-Gaussian nature of our model renders this filter sub-optimal. However, Sorenson and Alspach (1971) provide the optimal recursive formulae for obtaining the filter and predictive densities of the state vector in these circumstances, and a formula for computing the likelihood function. The recursive equations for computing the filtering and predictive densities are given in the form of integrals, whose analytical closed-form expressions are generally intractable, except in very special cases. In this paper, we numerically evaluate these integrals.

Using the homoskedastic state-space model we first establish the leptokurtic nature of the inflation data. We then investigate the Friedman hypothesis of a positive relationship between the level of inflation and its uncertainty. This is done by introducing conditionally heteroskedastic models, where the heteroskedasticity is modeled as contingent on the level of the state variable. This enables us to directly test the Friedman hypothesis with a standard likelihood ratio test. We demonstrate the relative sluggishness of Gaussian models in adapting to regime changes as compared to models with stable shocks. We also show that stable models rightly attribute greater uncertainty during confusing episodes, such as when outliers or regime shifts occur.

In Chapter 3 we investigate the CRSP returns series. We examine the data for evidence of leptokurtosis, and for seasonals in its mean and volatility. We investigate the persistence in return volatility with a GARCH-like model. To test for predictability of mean returns we set up a state-space model, with a persistent and predictable component in the conditional mean specification, driven by stable disturbances. We estimate the non-Gaussian state-space model using the Sorenson-Alspach (1971) recursions, as in Chapter 2, taking into account the seasonals and persistence in its volatility.

In Chapter 4 we investigate the U.S. real GNP series. We first examine the non-normality of its empirical distribution within a standard ARIMA framework, and the validity of the iid assumption in a univariate representation of this time series. We then extend the ARIMA model to incorporate asymmetries in the propagation of impulses.
The non-linear model has a simple and parsimonious representation, and nests the standard ARIMA model. Long memory is investigated by generalizing to fractional differencing. Impulse responses and measures of persistence are reported.

The final chapter summarizes the main results obtained in this study. It offers some broad conclusions on the empirical features of the three macroeconomic time series examined and the use of the time series models developed for describing these series. It also points out some limitations of this work and some directions for future research.

Appendix A provides some basic properties of stable random variables. Appendix B gives the Sorenson and Alspach (1971) general recursive filtering formulae, and provides details on the numerical integration procedures adopted in Chapters 2 and 3 for their implementation. Appendix C provides details on the inflation data used in Chapters 2 and 3, and its seasonal adjustment. Appendices D, E, and F contain the tables and figures summarizing the results obtained in Chapters 2, 3, and 4, respectively.
CHAPTER 2

OPTIMAL SIGNAL EXTRACTION WITH STABLE SHOCKS: THE CASE OF U.S. INFLATION

Summary: Most economic time series indicate non-normality in the form of either occasional big shocks or marked changes in the level of the series. In this paper, a univariate state-space model with infinite variance symmetric stable shocks is used to model the U.S. inflation rate via the Sorenson-Alspach (1971) recursions. Even after removing state-contingent heteroskedasticity, normality is rejected in favor of a stable distribution with exponent 1.81. Relative to the Gaussian model, the stable model accounts for outliers and level shifts better, provides tighter estimates of trend inflation, and gives more realistic assessment of uncertainty associated with it during confusing episodes.

2.1 Introduction

The state-space representation is a very powerful device for the analysis of time series. It provides a unifying framework for the analysis of a very wide variety of models, such as the ARIMA class of models, the unobserved components models, structural time series models, as well as non-linear and regression models. Much of its power is derived
from the fact that the Kalman filter provides a very efficient computational algorithm for the recursive estimation of the state vector. The filter also enables maximum likelihood estimation of any unknown hyperparameters in the system through the prediction error decomposition form of the likelihood function.

The Kalman filter delivers the minimum mean square error estimator of the state vector when the disturbances and the initial state vector are normally distributed. When this assumption does not hold, but variances are finite, the estimators of the state vector are optimal only within the class of linear estimators, and are not guaranteed to be the means of the conditional distributions of the state vector (see Harvey (1992), p.105).

The assumption of normality is commonly motivated in econometrics on the basis of the central limit theorem and the fact that much of the economic data is aggregated across time and across agents. Yet, there are several instances where the data clearly reveal obvious departures from normality. For example, Blanchard and Watson (1986) argue that evidence of excess kurtosis found in the residuals of a structural VAR that includes aggregate prices, output, money and a fiscal policy variable points to large infrequent shocks in the series. Balke and Fomby (1994) examine fifteen postwar U.S. macroeconomic time series for outliers using univariate autoregressive models and find significant excess kurtosis in all but one of the series (the money base), indicating the presence of outliers. They find that outliers account for more than 20 percent of the volatility in many of their series, with the percentage exceeding 50 for the GNP deflator, the CPI, nominal compensation and M2. It is now well-known that stock returns exhibit a level of kurtosis in excess of what is consistent with normally distributed returns. The big decline in U.S. real GNP during the Great Depression could easily be accommodated within the probabilistic assumption of heavy-tailed distributions. A second feature often observed in economic data is abrupt regime changes over different sub-periods. For instance, a cursory look at inflation and nominal interest rates indicates marked changes in the behavior of the series over different sub-periods.

The Generalized Central Limit Theorem delivers the result that the sum of iid random variables must have a limiting distribution from the stable family if such a
limiting distribution exists (Zolotarev (1986, Ch.1)). Thus, the very attribute that makes
the normality assumption so appealing in economics motivates the case for considering
more general stable shocks (which include the normal as a special case) in modeling
economic series. While outliers can also be accounted for by assuming alternative fat-
tailed distributions such as the \( t \)-distribution, and regime changes are often modeled in the
literature using mixtures of normals or Markov regime switching models, these statistical
distributions cannot arise from aggregates of random variables.

The non-Gaussian members of the stable family all have thicker tails than the
normal density. In fact for all but the normal distribution, the second moment does not
exist (see Appendix A for some basic properties of stable random variables). Stable
models have been estimated for stock returns by Fama (1965), Buckle (1995), and
Mantegna and Stanley (1995), and for interest rate movements by McCulloch (1985), and
Oh (1994). See McCulloch (1996b) for additional references.

The estimation of non-Gaussian state-space models presents a problem as the
Kalman filter no longer provides the posterior distributions of the state vector. However,
recursive formulae for obtaining the filter and predictive densities of the state vector in
the general case have been derived by Sorenson and Alspach (1971). The relevant
densities may be evaluated numerically where these are analytically intractable.
Estimation of hyperparameters is feasible by maximum likelihood. The mean of the filter
density gives the expectation of the state variable conditional on the data observed to
date.

The present study demonstrates the feasibility of modeling macroeconomic time
series with symmetric stable shocks by means of the Sorenson-Alspach filtering
formulae. For this purpose we univariately model the monthly, CPI-X adjusted U.S.
adjustment of the series are given in Appendix C.

The univariate model developed in this paper can be used for forecasting of
inflation and to simulate historical inflation forecasts conditional on the history of
inflation. It may also serve as a useful starting point for the construction of more complex multivariate models.

Section 2.2 below implements the recursive nonlinear filter for a simple local level model of inflation with homoskedastic stable shocks. Section 2.3 refines the model to include state-dependent heteroskedasticity in the disturbances. Section 2.4 investigates the implications of the stable model for inflationary expectations conditional on realized inflation. Section 2.5 concludes and briefly discusses possible extensions of this work.

2.2 A Homoskedastic Local Level Model

To begin with we consider a simple local level model for inflation as follows:

Model 1

\[
\begin{align*}
y_t &= x_t + \varepsilon_t, \\
x_t &= x_{t-1} + \eta_t,
\end{align*}
\]

where \( S_\alpha(0,c) \) refers to a symmetric stable distribution with characteristic exponent \( \alpha \), scale parameter \( c \), and location \( \delta = 0 \). Here, \( y_t \) is the seasonally adjusted observed inflation, and \( x_t \) is an unobserved trend inflation. The \( \varepsilon_t \) shock captures transitory deviations of observed inflation about the trend, and \( \eta_t \) captures permanent shocks that drive the trend inflation. The two shocks are assumed to be serially and mutually independent. When viewed within the framework of state-space models, \( x_t \) is the state variable, \( \varepsilon_t \) is the measurement error, and \( \eta_t \) is the signal shock.

In the Gaussian case \( \alpha = 2 \), the local level model is equivalent to an IMA (1,1) process, with the first-order autocorrelation coefficient restricted between zero and minus one-half. A number of studies, including Barsky (1987), Ball and Cecchetti (1990), and Brunner and Hess (1993), identify such a process as describing the time series of U.S. inflation rates, to at least a first approximation. Nelson and Plosser (1982) identify IMA(1,1) processes as being appropriate for describing most of the fourteen macroeconomic series that they study.

For \( \alpha < 2 \), a subtle distinction emerges between the local level and IMA(1,1) models. The present study adheres to the former model, because of its natural
interpretability as a slowly changing regime observed with noise. In doing so, however, we are validly extending the Gaussian IMA(1,1) literature to the non-Gaussian stable cases.

In the Gaussian case, the Kalman filter provides optimal, linearly adaptive estimates of $x_t$ and therefore of future $y_t$ in (2.1) above. However, if $\alpha < 2$, the Kalman filter does not provide optimal estimates. Nevertheless, the filtering algorithm of Sorenson and Alspach (1971) enables estimation of both the likelihood function as well as the conditional densities of the state vector under general assumptions about the distributions of the disturbances.

Let $Y_t = (y_1, \ldots, y_t)$. The recursive formulae for obtaining one-step ahead prediction and filtering densities, due to Sorenson and Alspach, are as follows:

$$p(x_t|Y_{t-1}) = \int_{-\infty}^{\infty} p(x_t|x_{t-1})p(x_{t-1}|Y_{t-1})dx_{t-1}, \quad (2.2a)$$

$$p(x_t|Y_t) = p(y_t|x_t)p(x_t|Y_{t-1}) / p(y_t|Y_{t-1}), \quad (2.2b)$$

$$p(y_t|Y_{t-1}) = \int_{-\infty}^{\infty} p(y_t|x_t)p(x_t|Y_{t-1})dx_t. \quad (2.2c)$$

Finally, the log-likelihood function, conditional on the hyperparameters $\alpha$, $c_\varepsilon$ and $c_\eta$, is given by:

$$\log p(y_1, \ldots, y_T) = \sum_{t=1}^{T} \log p(y_t|Y_{t-1}). \quad (2.3)$$

These formulae have been applied to non-Gaussian data and extended to include a smoother formula by Kitagawa (1987). Details concerning the numerical implementation procedure used in this chapter are given in Appendix B.2. When $\alpha = 2$, this filter collapses to the Kalman filter.

Maximum likelihood (ML) estimates of the hyperparameters $\alpha$, $c_\varepsilon$ and $c_\eta$, using monthly U.S. inflation data for 1953:11 (Oct-Nov inflation) to 1993:9 ($T = 479$), are given in the first row of Table D.1 as Model 1-S. The estimate of the characteristic exponent $\alpha$ is 1.803, well below the value of 2 for normal shocks. The scales imply a
signal-noise scale ratio of 1:4.9 suggesting a relatively smooth trend inflation as compared to the erratic observed series at this monthly frequency. Hessian-based asymptotic standard errors are in parentheses.

We also estimated the above Model, constraining $\alpha = 2$, i.e. assuming normal distributions for the error terms. In this case the Kalman filter was employed for estimation. The resulting ML hyperparameters are given in the first row of Table D.2 as Model 1-N. The estimated scales reported in Table D.2 (which for $\alpha = 2$ equal standard deviation divided by $\sqrt{2}$) imply a first-order autocorrelation coefficient of -0.49. The estimated scales imply a signal-noise scale ratio of about 1:4.4, or a variance ratio of 1:19.4.

The last column of Table D.2 shows the likelihood ratio (LR) statistic $2\Delta \log L$ for the restriction $\alpha = 2$ to be 24.76 in Model 1. This LR test statistic does not have the usual $\chi^2(1)$ distribution since the null hypothesis lies on the boundary of admissible values for $\alpha$ and, hence, the regularity conditions are not satisfied. However, McCulloch (1996a) has tabulated small sample critical values for such a test by Monte Carlo simulations in the simple case when $\alpha$, $c$ and $\delta$ are being estimated. The models in this paper are more complicated, but this is unlikely to greatly affect the critical values for testing $\alpha = 2$. Based on these critical values, the likelihood ratio test easily rejects normality in favor of $\alpha < 2$ at the 0.005 level of significance (critical value 7.664 for $T = 300$, 6.688 for $T = 1000$), under the homoskedastic assumption of Model 1. A Wald statistic computed from the asymptotic standard error on $\alpha$ likewise does not have an asymptotic normal distribution for the null hypothesis $\alpha = 2$, and, indeed, it greatly underpredicts the actual change in log-likelihood.

A plot of the standardized residuals (not shown) indicates neither any apparent deviation from a zero mean process nor any trending behavior, and the Box-Ljung test reveals no significant residual serial correlation. The asymptotic distribution of the Box-Ljung test statistic as a Chi-square requires a finite second moment assumption (see Ljung and Box (1978), and Anderson and Walker (1964)). Therefore, although the inferences from this test hold asymptotically in the Gaussian case, these can only be
taken as suggestive when shocks are non-normal stable. For the model with normal
shocks, the Goldfeld-Quandt test statistic for homoskedasticity has a value of 1.174 with
a p-value of 0.157 when the residuals are sorted sequentially by their occurrence in time.
One should note, however, that chronological sorting of residuals could easily mask
clustering of big shocks (ARCH effects) or even a dependence of variances on the trend
inflation. When the test is repeated by sorting the residuals by the absolute value of $x_t$,
the test statistic rises to 1.522 with a p-value of 0.004. With stable shocks, since the
variances are infinite, the Goldfeld-Quandt test is not directly applicable. However, the
results from the Goldfeld-Quandt test may be regarded as suggestive of
heteroskedasticity in the stable case too. Below, we formally test for homoskedasticity
with nested hypothesis testing procedures under both normal and stable shocks.

Although Engle (1983) finds ARCH effects in U.S. quarterly inflation from
1947.IV to 1979.IV, Cosimano and Jansen (1988), using the same data set, show that
Engle’s model is misspecified due to the presence of residual serial correlation and
demonstrate that, when the model is respecified so as to eliminate the autocorrelations,
not only do the ARCH effects disappear but the conditional variance of inflation is
constant. Similarly, So and Edmonds (1995) find no evidence of any statistically
significant ARCH effects in Engle’s model for U.S. inflation measured at monthly,
quarterly and annual frequencies. However, other studies (e.g., Bollerslev (1986),
Brunner and Hess (1993)) have documented heteroskedastic behavior in inflation. In any
case one should note, as Brunner and Hess also point out, that ARCH/GARCH processes
could proxy for state-level-dependent heteroskedasticity due to persistence in inflation.

2.3 The Inflation-Uncertainty Relationship

The relationship between the level of inflation and its variability has been
extensively investigated in empirical research. Early studies, focusing on cross-country
relations, found that countries with high inflation rates also experienced more variable
inflation. Later work using both cross-country and time series data also reported a
positive relationship between the level of inflation and its estimated variance. However,
Engle’s (1983) study of U.S. inflation rates does not reveal any significant relationship
between the conditional variance and the average level of inflation. Subsequent work by Cosimano and Jansen (1988) confirms this finding.

Ball and Cecchetti (1990) decompose observed inflation into a trend component capturing long-term movements in inflation and stationary deviations about this trend, and find that a high level of trend inflation leads to greater variability of both components. However, they emphasize that high inflation increases uncertainty relatively more in the long run than in the short run. This differential effect of inflation rate on the uncertainty at different horizons has also been documented by Evans and Wachtel (1993) within a framework which allows them to explicitly model different inflation regimes during their sample period. This has been recognized to be important, as tests for parameter stability of inflation processes frequently fail (Engle(1983), Cosimano and Jansen (1988)). Evans and Wachtel incorporate regime dependence of the inflation process within a Markov switching model that allows them to decompose total inflation uncertainty into a component arising from future inflation shocks and another arising from uncertainty about the inflation regime, and find that at longer horizons variations in the regime uncertainty component track movements in the rate of inflation.

While the Markov switching model can account for discrete shifts in a process it admits only a finite number of states, with computational tractability limiting these to a small number. It further implies that inflation is driven by a complicated mixed stochastic process. A more appealing framework to account for the occasional abrupt changes in the mean value of the rate of inflation is a stochastic process driven by shocks drawn from continuous yet thick-tailed distributions. This results in a simpler, flexible and more parsimonious representation that can capture abrupt changes quite easily, while retaining the advantages of continuity.

In this section we examine the relationship between the level of inflation and uncertainty associated with its future behavior within the framework of univariate stable stochastic processes. In particular we also investigate the differential effect of inflation on uncertainty at different horizons. Our statistical models permit direct examination of
these relationships because we model conditional variability as functions of the level of inflation rate.

A convenient way of modeling links between the level of inflation and its variation is to introduce a dependence of the scales of the two shocks on the trend level of inflation, as in Ball and Cecchetti (1990) and Brunner and Hess (1993). Here, we use a slight variant of their formulation as follows:

Model 2

\[ y_t = x_t + \epsilon_t, \quad \epsilon_t \sim S_\alpha(0, c_{\alpha \epsilon}), \quad c_{\alpha \epsilon}^\alpha = c_{\epsilon 0}^\alpha (1 + \theta_\epsilon |x_t|) \]  

(2.4a)

\[ x_t = x_{t-1} + \eta_t, \quad \eta_t \sim S_\alpha(0, c_{\eta \eta}), \quad c_{\eta \eta}^\alpha = c_{\eta 0}^\alpha (1 + \theta_\eta |x_{t-1}|) \]  

(2.4b)

We restrict \( \theta_\epsilon \) and \( \theta_\eta \) to be non-negative. Statistically significant estimates of \( \theta_\epsilon \) and/or \( \theta_\eta \) would validate the hypothesis that a higher level of trend inflation leads to greater uncertainty. On the other hand, if there is no such relationship then we should obtain estimates of the \( \theta \)'s that are indistinguishable from zero. This formulation also allows us to investigate the main contention in Ball and Cecchetti's paper that the trend level of inflation affects the variability of the permanent shocks more than it affects the variability of the temporary shocks, i.e. \( \theta_\eta > \theta_\epsilon \).

Even with normal shocks, the maximum likelihood estimation of Model 2 is not feasible with the Kalman filter due to the state-dependency of the scales. Ball and Cecchetti sidestep the direct estimation of such a model, citing complicated econometrics. Instead they take the simpler approach of estimating cross-country relations between the level of inflation and its variance, and within-country relations by breaking up their data into five-year periods and estimating this relation across the different sub-periods.

However, the general recursive filtering algorithm (2.2a)-(2.2c) can once again be usefully applied for estimating this model. We estimated the above model by maximum likelihood, both with \( \alpha \) unrestricted (Model 2-S in Table D.1) and with \( \alpha \) restricted to 2 (Model 2-N in Table D.2). Although the asymptotic standard errors do not indicate that either \( \theta_\epsilon \) or \( \theta_\eta \) is significantly different from zero, the LR statistic for Model 1 vs.
Model 2 in Table D.3 strongly rejects the hypothesis that they are both zero, whether or not normality is imposed.

Models 2a and 2b respectively set $\theta_{\varepsilon}$ and $\theta_{\eta}$ to zero. The latter is rejected by the LR test in Table D.3 for both the stable and Gaussian models, despite the large asymptotic standard errors on the Model 2 $\theta_{\eta}$ estimates. Model 2a is not rejected by the LR test in the stable model, but is in the Gaussian model, again despite a large asymptotic standard error on the $\theta_{\varepsilon}$ estimate.

Model 2 may be simplified by setting $\theta_{\varepsilon} = \theta_{\eta}$ to obtain Model 3. This implies that the state variable contributes equiproportionately to the variability of the two shocks. Although there is no theoretical justification for this restriction, it makes the model more parsimonious and results in a constant signal-noise ratio. The results indicate a small reduction in the maximized log-likelihood value over Model 2-S. However, both the AIC and Schwarz criteria rank the restricted model over the general version, due to its more parsimonious representation. An LR test rejects the hypothesis $\theta_{\varepsilon} = \theta_{\eta}$ at 5 percent.

Models 2a and 3 are not nested, but 2a has the higher likelihood with the same number of parameters. A test of Model 2a-S versus Model 3-S fails to reject Model 2a. Therefore among the four possible models designed to capture heteroskedasticity in inflation, Model 2a-S seems to work best.

The exact form of the heteroskedasticity is not so strongly identified by the data as is its presence. The frequent divergence between the LR statistics in Table D.3 and the squared Wald statistics implied by Tables D.1 and D.2 indicate that the likelihood surface diverges from a quadratic shape that would make inferences about the form of heteroskedasticity more clear-cut. The estimated value of $\alpha$ is about 1.81 in all models. The estimate of $\theta_{\eta}$ in the preferred Model 2a-S, 1.400, implies that at 10 percent trend inflation, the scale of the signal shocks is 1.08 times as high as at zero trend inflation. However, the effect of this is empirically obscured by the much larger noise errors, which may be taken as homoskedastic.
In order to determine which model would result if we worked with normal shocks throughout, we estimate all the above models assuming normality for the \( \varepsilon \) and \( \eta \) shocks. The estimation results are reported in Table D.2. The likelihood ratio test results in Table D.3 show that with normal errors Model 3 is identified as being most suitable in terms of the AIC. Note, however, that the estimated values of the parameters imply that the scales are not greatly affected by the level of trend inflation. For instance, the scale for the normal Model 3 at an average annual trend inflation of 10 percent is only about 1.006 times as large as it is at zero inflation.

It is conceivable that the rejection of normality in the homoskedastic Model 1 was driven by conditional heteroskedasticity proxying for leptokurtosis. The last column of Table D.2 however shows that the LR statistic strongly rejects \( \alpha = 2 \) for all Models 2 through 3. Thus inflation appears to display strong leptokurtosis that is not a result of state-dependent conditional heteroskedasticity.

2.4 Effects of Non-Normality on Inflation Forecasts

It can be shown that so long as \( \alpha > 1/2 \), the filter density has finite variance for \( t \geq 2 \), despite the infinite variances of the stable shocks. Table D.4 gives the optimal inflation forecast from stable Model 2a-S, \( E(x_t|Y_t) = E(y_{t+j}|Y_t) \), \( j \geq 1 \), along with its numerically computed standard error. Figure D.1 plots the complete filter density every twelfth month.

Figure D.2, panel A plots the observed inflation series along with the filter mean and a 2-standard error band derived from Table D.4. For comparison, panel B plots the same information from the preferred Gaussian Model 3-N. Divergences between the two sets of forecasts are particularly important during three episodes: 1956-7, 1973-4, and 1985-6, which we will examine in detail below.

In Figure D.3 we plot the standard errors computed from the filter densities from the stable model 2a-S, along with those from the Gaussian model 3-N. Without any heteroskedasticity, the standard errors from the normal model (as in Model 1-N) would be constant except for the startup of the filter. The variation in the normal standard errors
that does appear after the startup is entirely due to the level of trend inflation - being high when trend inflation is high and low at other times. The normal model standard errors, like those of a standard Kalman filter, do not respond at all to the magnitude of the local disturbances, except indirectly through the state variable.

Because the stable model makes more efficient use of the data (see Batchelor 1981), its standard errors are usually smaller than those from the normal model. It, therefore, ordinarily gives more precise forecasts of future inflation, reacting to variation in the state variable in a similar way, but at a lower level.

During unsettled periods, when large outliers or regime shifts appear to have occurred, however, the stable filter density spreads out and, as we shall see, may even become multimodal. During these periods, the stable forecasts appropriately have higher standard errors. Because the normal filter density does not respond directly to the local disturbances, it must overcompensate during quiet periods with an unnecessarily high standard error.

In Figure D.4, panel A we plot the posterior densities of trend inflation from the two models during the unsettled period 1956:3-1957:2 when inflation rose abruptly from its prior level. The dashed vertical lines indicate the actual observed inflation during the specific month. Before 1956:4 trend inflation had been around zero or even slightly negative. In April both models predicted about zero inflation, the stable forecast being more concentrated. Then, three big shocks, corresponding to observed inflation in May, June and July of about 5.7, 6.2, and 8.5 percent respectively, were realized in succession. The figures show the different response of the normal and stable models to these shocks. Specifically, the stable model mostly ignores the first big shock whereas the normal model reacts with a substantial jump in the mean. With the second big shock, the stable model begins to show mass in the right tail to accommodate the possibility that a large level shift has occurred, whereas the normal model undergoes a further shift of its entire distribution. With the third shock, however, the stable model has adapted (and more than the normal model) to what it perceives as a level shift with most of the probability mass concentrated close to the newly observed inflation. However, it still has a smaller mode
back near zero (corresponding to the old mode) just in case these are just three unusual draws of big $\varepsilon$ shocks in succession. The next two months suggest that this in fact was true and the stable model is quick to revert back to the old mode whereas the normal model is slower to react. By March of 1957, both models have settled down on an estimate of 2-3 percent trend inflation.

Figure D.4, panel B illustrates the response of the two models to one very big shock that occurred in August of 1973. The figures in the first row show that both models estimated trend inflation to be around 6-8 percent prior to this period. The observed seasonally adjusted inflation for August was nearly 20 percent - a jump of about 14 percentage points. The figures show that the Gaussian model reacts strongly with a large shift in location (from about 6 percent to about 9 percent) whereas the stable model simply develops a small mode far out in the tail. Observed inflation in the next period was about 1 percent. The stable model now loses its small new mode in the right tail completely, treating the big shock as a clear outlier, while the Gaussian model continues to forecast over 8 percent inflation. Both models subsequently concur that trend inflation was on the rise.

Panel C in Figure D.4 tells a story similar to the first. Starting from a period of moderate inflation of about 3.5 to 4.0 percent, three fairly large negative shocks were observed in succession. Seasonally adjusted inflation for February, March and April 1986 was -3.9, -5.8, and -3.0 percent, respectively. The Gaussian model reacts sluggishly. On the other hand, the stable model develops a very small mode at the first shock, which builds into a much bigger mode at the second shock (while retaining the old mode), and has almost fully adapted to a possible level shift by the third. But when the following observations fail to confirm a level shift, the stable model is quick to revert to the old mode.

Thus, these three panels in Figure D.4 illustrate two important differences in the behavior of stable and Gaussian processes. First, when faced with occasional big shocks the Gaussian model reacts strongly with a big jump in the mean whereas the stable model ignores these, treating them instead as outliers. Second, when consecutive shocks of
similar magnitude are encountered the normal model is very sluggish in fully adapting, whereas the stable model infers that the regime has changed more quickly. A further fact that emerges from these figures is that during such episodes the stable model more realistically gives higher estimates of uncertainty associated with trend inflation. This is understandable since the stable model usually develops bimodal densities reflecting greater uncertainty during these confusing periods whereas the normal model does not.

Abstracting from heteroskedasticity and startup considerations, the Gaussian Kalman filter density is always normal, with a constant standard error, and adapts linearly to new information. Stuck (1978) and Rutkowski (1994) find the optimal stable filter within the linear class for discrete time processes. Similarly, Le Breton and Musiela (1993) generalize the Kalman filter to continuous time processes with and without infinite variance errors. However, the mean of the globally optimal filter does not revise linearly unless the shocks are truly Gaussian.

The level of future inflation, as implied by the local level model, is:

$$y_{T+j} = x_T + \sum_{i=1}^{j} \eta_{T+i} + \epsilon_{T+j}$$  \hspace{1cm} (2.5)

In the homoskedastic Model 1, the stability property of the $\eta$ and $\epsilon$ shocks implies that $(y_{T+j} - x_T) \sim S_\alpha(0, c_j)$ where $c_j = (jc_\eta^\alpha + c_\epsilon^\alpha)^{1/\alpha}$. Since $x_T$ is unobserved and has to be estimated, the forecast density of future inflation is obtained by convoluting $S_\alpha(0, c_j)$ with the filter density of $x_T$. Thus, $c_j$ provides a lower bound on the uncertainty associated with forecasting future inflation. For large values of $j$, $c_j$ is approximately $j^{1/\alpha}c_\eta$, and the contribution of the filter density is small.

Similarly, the future price level, $\ln(p_{T+j})$, is given by:

$$\ln(p_{T+j}) = \ln(p_T) + jx_T + \sum_{i=1}^{j} \sum_{k=1}^{i} \eta_{T+k} + \sum_{i=1}^{j} \epsilon_{T+i}$$  \hspace{1cm} (2.6a)

$$= \ln(p_T) + jx_T + \sum_{k=1}^{j} (j-k+1) \eta_{T+k} + \sum_{i=1}^{j} \epsilon_{T+i}$$  \hspace{1cm} (2.6b)
The stability property of the errors again implies that:
\[ \{ \ln(p_{T+j}) - \ln(p_T) - jx_T \} \sim S_\alpha(0,C_j) \]
where \( C_j = \{(j^\alpha + (j-1)^\alpha + \ldots + 1)c_\eta^\alpha + jc_e^\alpha\}^{1/\alpha} \). Once again this density needs to be convoluted with the filter density of \( jx_T \) in order to obtain the forecast density of the future price level. For large values of \( j \), \( C_j \) behaves approximately like \( \{j^{(\alpha+1)} / (\alpha + 1)\}^{1/\alpha} c_\eta \), while the scale of \( jx_T \) grows only in proportion to \( j \).

For Models 2 and 3, the exact forecasting uncertainty can only be found by a series of convolutions, taking the heteroskedasticity into account. Nevertheless, it may be expected that the above formulae, using the Model 1 estimated parameters, give a rough approximation of the true forecasting uncertainty.

### 2.5 Conclusions and Extensions

The present paper demonstrates, using U.S. inflation data, that it is feasible to numerically implement the nonlinear filtering algorithm of Sorenson and Alspach (1971) with non-Gaussian stable disturbances. Even after adjusting for state-contingent heteroskedasticity, normality is strongly rejected in favor of stable distributions with characteristic exponent \( \alpha \) equal to 1.81. The mean of the filter density for trend inflation is the optimal univariate forecast of future inflation under the postulated local level model.

The method can readily be extended to other stationary or non-stationary time series that exhibit leptokurtosis, such as stock returns, real income growth, and real interest rates. Kitagawa (1987) develops a recursive formula for the smoother density \( p(x_t|Y_T) \), which may be useful in these contexts. The present study focuses on simulated inflation forecasts, and therefore does not compute the Kitagawa smoother.

It would be interesting to investigate how state estimates from stable processes compare with those obtained from Markov regime switching models (Hamilton (1989), Lam (1990)). The assumption of stable shocks and the assumption of regime switching components can be viewed as alternative modeling schemes to account for outliers or
occasional shifts in a time series. While the regime switching models limit the number of states to a discrete few (computational burden severely restricts the number of states in these models), the stable models can account for a continuum of state values and are therefore capable of more realistically capturing the varying degrees of shifts that institutional or policy changes may effect.

Throughout this study we assumed that shocks are symmetrically distributed. However there is some evidence of skewness in economic series. For instance, Buckle (1995) finds that stable asset returns are positively skewed. Balke and Fomby (1994) find significant skewness in many of the fifteen postwar U.S. macroeconomic time series they examine. At present it is much more difficult to work with skew-stable shocks, since no fast numerical approximation to asymmetric stable densities has been developed.

Furthermore, our model for heteroskedasticity does not consider any asymmetries in the effects of the inflation rate on its uncertainty. Such asymmetries in the conditional variance of inflation have been found to be significant by Brunner and Hess (1993). These could be captured by state-dependent models of conditional moments, as Brunner and Hess consider, or by EGARCH.

Our methodology, based on univariate models, suffers from the drawbacks of investigating economic signals in isolation. It limits exploration of interesting interactions between different macroeconomic series. Computational expense and instabilities in current high-dimensional numerical integration techniques at present discourage implementation of the Sorenson-Alspach (1971) filtering technique for state-space models with multiple elements in the state vector. However, Monte Carlo integration techniques may ultimately overcome these difficulties.
CHAPTER 3

REAL STOCK RETURNS: NON-NORMALITY, SEASONALITY, AND VOLATILITY PERSISTENCE, BUT NO PREDICTABILITY

Summary: We investigate persistence in CRSP monthly real stock returns, using a state-space model with symmetric stable disturbances. The non-Gaussian state-space model is estimated by maximum likelihood, using the optimal filtering algorithm given by Sorenson and Alspach (1971). Volatility seasonals and volatility persistence are quite strong. The conditional distribution has a stable $\alpha$ of 1.89, and non-normality is strongly rejected. However, stock returns do not contain a significant mean-reverting component. The optimal predictor is the unconditional expectation of the series, which we estimate to be 9.2 percent per annum.

3.1 Introduction

The predictability of stock returns has received an enormous amount of research attention (see Fama (1991) for a recent survey). While weak evidence on short-horizon predictability has been acknowledged for some time now by many researchers, there is great disagreement on long-horizon predictability first reported by Fama and French
(1988) (see, for instance, Richardson and Stock (1989), Kim, Nelson and Startz (1991), McQueen (1992), etc.). If stock returns truly contain a persistent signal, obscured by random noise, an unobserved components model provides a convenient description of the observed series as the sum of an unobserved signal and a random noise.

Such unobserved components, or state-space, models enable recovery of the persistent signal from their observed noisy indicator with minimal auxiliary assumptions. State-space models have been used to study real income series by Harvey (1985), Watson (1986), and Clark (1987), and bond returns by Oh (1994). Most applications of state-space models, including Harvey (1985), Watson (1986) and Clark (1987), assume normality of the underlying error disturbances. This assumption is primarily motivated by the ease of estimation that the powerful Kalman filter affords under this distribution.

However, non-normality of stock returns has been widely noted by a number of authors, the more recent among these being Akgiray and Booth (1988), Jansen and de Vries (1991), Buckle (1995), Mantegna and Stanley (1995), and McCulloch (1996a,b). Stable distributions have been used to describe stock returns by a number of studies, including the last three of these. In this paper, we investigate whether or not a persistent predictable signal is present in the CRSP value-weighted monthly real returns, using a state-space model with stable shocks. Our model should enable more efficient estimation of the persistent component if such a component exists and stock returns are truly non-Gaussian.

There is a large literature that provides empirical evidence on the persistence of stock return volatility (see, for instance, Nelson (1991), Danielsson (1994), Pagan and Schwert (1990), Diebold and Lopez (1995), Ghose and Kroner (1995), etc.). Accurate description of stock return volatility is important to efficiently measure any persistence in mean returns that may be present. Therefore, we investigate volatility persistence in some depth in this paper.

Fama (1991) gives a discussion of seasonal components in stock returns. We explicitly investigate seasonal effects in both the mean and the volatility of stock returns.
Our stock returns data are derived from the monthly value-weighted CRSP returns series with dividends. The data spans the period starting from February 1953 through December 1994. We convert the nominal arithmetic returns to continuously compounded logarithmic returns. Inflation is known to have strong persistence (see Chapter 2), so these are then deflated by monthly CPI-based inflation to obtain real returns. Figure E.1 plots this series.

This chapter is organized as follows. In section 2 we establish the strong departure of CRSP returns data from iid normal behavior. In Section 3 we investigate seasonals in the mean and volatility of returns. The persistence in volatility is examined in section 4, and the persistence in mean returns in section 5. Section 6 offers some concluding remarks. Appendix A provides some basic properties of stable random variables.

3.2 Departures from normality

It has been well documented in the literature that the empirical distribution of stock returns exhibits a higher degree of kurtosis than is consistent with normal densities. The Generalized Central Limit Theorem points to the stable distributions as the natural extension of the normal to account for the thick tails, but Student-$t$ distributions, mixtures of normals, and the Weibull distribution have also been proposed.

Lack of consensus on how best to model stock returns arises from disagreement on the amount of probability mass in the tails. This line of research has resulted in criticism of stable distributions as being appropriate for modeling returns. For instance, researchers (e.g. Jansen and de Vries (1991)) have argued that, although stock returns data show excess kurtosis compared to normal densities, they still possess finite second

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1 Our measure of inflation is based primarily on the CPI-U. However, the CPI-U is generally recognized to have seriously mismeasured the housing component of the true cost of living prior to its 1983 revision. The Bureau of Labor Statistics (BLS) CPI-X series recomputes the CPI-U on the post-1983 basis and is available from June 1967 until its adoption as the official CPI-U in 1983. We therefore constructed a CPI-UX series by linking together the CPI-U from 1953:1 to 1967:6, the CPI-X from 1967:6 to 1983:1, and again the CPI-U from 1983:1 to 1994:12. The 1967 base year was used for both series, so as to reduce rounding error. During 1967:6 to 1983:1, the CPI-U and CPI-X diverge by 8.5 percentage points. The inflation rate was constructed as the first differences of the natural logarithms of the constructed CPI-UX series (see Appendix C).
moments and, hence, have tails thinner than those of infinite variance stable shocks. Mantegna and Stanley (1995), therefore, suggest modeling the center of the distribution as a stable law, with tails being exponential, thereby keeping the variances finite.

Studies such as Akgiray and Booth (1988), and Jansen and de Vries (1991), focusing exclusively on the tail behavior of returns, typically report estimates of the Paretian exponent of the tails in excess of the value of 2, which is the upper bound for infinite variance stables. Ghose and Kroner (1995) provide Monte Carlo evidence suggesting that the estimates of the tail indices for data generated from conditionally normal IGARCH models are always greater than 2. They estimate a value of 3.54 for the tail index for the SP500 series, and conclude that these returns are better characterized as IGARCH-normal processes rather than iid stable processes. However, based on the Monte Carlo simulation evidence of McCulloch (1996a), it is not surprising to obtain estimates of the tail index in excess of 2, when the data are drawn from stable distributions with characteristic exponent as low as 1.65. Therefore, rejection of stable models based on such asymptotic estimates of the tail index is invalid.

If stock returns are truly iid stable stochastic processes, then one expects to find identical estimates of the characteristic exponent for the returns data sampled at daily, weekly, and monthly frequencies. Failure to obtain such matching estimates has been cited as evidence against stability (Akgiray and Booth (1988)). Furthermore, Ghose and Kroner (1995) show that the estimate of the characteristic exponent may remain constant under temporal aggregation even for data generated by an integrated GARCH (IGARCH) model. Therefore, these authors argue that even if returns were to exhibit this property, one cannot conclude that these series are stable. However, differing estimates of the characteristic exponent based on data sampled at different frequencies constitute a rejection of iid stability. Seasonals in the mean and/or volatility of returns, and the temporal clustering of return shocks, could lead to non-identical and non-independent returns. Therefore, rejection of iid stability could very well be driven entirely by the invalidity of the iid assumption. That the volatility process may actually be IGARCH in no way rules out conditionally stable shocks. McCulloch (1985) estimates such a process
for bond returns. Thus, the empirical objections above in no way weaken the case for modeling stock returns as stable stochastic processes.

To investigate the non-normality of stock returns, we begin by estimating the following two preliminary models by maximum likelihood (ML):

\[ r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim \text{iid } N(0, \sigma^2) \]  
(3.1)

\[ r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim \text{iid } \text{S}_\alpha(0, c) \]  
(3.2)

where \( \text{S}_\alpha(0, c) \) indicates a symmetric stable distribution with characteristic exponent \( \alpha \), location zero, and scale \( c \). The above models assume that returns, \( r_t \), are unpredictable, apart from a constant mean, \( \mu \).

The estimation results are reported in Table E.1. Results for Gaussian model (3.1) indicate an average return of about 0.56 percent per month. The ML estimator rises to 0.68 percent per month in the stable model (3.2), with an estimated value of \( \alpha \) of 1.833. The table also shows a large increase in the log-likelihood value of the stable model over the normal model. The likelihood ratio (LR) test statistic for \( \alpha = 2 \) has a non-standard distribution, since the null hypothesis lies on the boundary of admissible values for \( \alpha \), and, hence, the standard regularity conditions are not satisfied. The small-sample critical values for such a test have been tabulated in McCulloch (1996a). Table E.1 reveals that \( \alpha = 2 \) is easily rejected at better than the 0.005 level based on these critical values. The improvement in log-likelihood of the stable model over the normal model stems from the ability of the transnormal stable shocks to capture the observed leptokurtosis in the returns series. The above test, however, without any provision for seasonal components or volatility persistence, provides only preliminary evidence against normality since, as will be shown below, the assumption of iid returns is not really valid.

3.3 Seasonals

Two important anomalies regarding seasonals in monthly mean returns have been observed in the literature. First, returns of small firms in January are higher than the returns of all other firms in the same month. Second, returns in January have been found to be higher on average than the returns in the rest of the year for the same stocks, this being true of returns on all equally-weighted size-based decile portfolios, except the
largest decile. Seyhun (1993) shows that these monthly seasonals are not only statistically significant, but also robust to the choice of the sample period over 1926-91, to the presence of outliers in the returns series, and to the higher transactions costs for dealing with smaller firm stocks. Various explanations, both consistent with the efficient markets hypothesis and equilibrium asset pricing models, and indicating failure of these, have been offered for the presence of these seasonal components. These explanations range from the year-end tax loss selling and portfolio rebalancing, to seasonalities in the risk-return tradeoff, omitted risk factors, and risk mismeasurement problems. Seasonals in return volatility may also be present.

Plots of the returns each month over the sample period (not shown) failed to reveal any significant seasonal patterns in the CRSP value-weighted real return series. In particular, no conspicuous behavior in the January returns was observed. Also, sample monthly averages failed to rank January as the month with the highest mean return.

A general model to account for potential seasonals in both the mean and volatility of returns is:

$$r_t = \mu + \sum_{i=1}^{12} s_i I_{ii} + \varepsilon_t, \quad \varepsilon_t \sim S_{0}(0, c_t)$$

(3.3)

with $I_{ii} = 1$ if $t$ corresponds to month $i$ and zero otherwise, seasonals $s_i$ sum to zero, i.e. $\sum_{i} s_i = 0$, and scales, $c_t$, which depend on the month, but otherwise are constant. The LR test for the non-seasonal model (3.2) versus the seasonal model (3.3) rejects the former at better than the one percent level ($2\Delta \log L = 42.23, \chi^2_{22} = 40.29$ at the 0.01 level). This indicates that at least some seasonals, in the mean and/or the volatility of real returns, are significant.

To isolate the seasonals further, we consider two constrained versions of model (3.3). One version allows for seasonals only in mean returns and the other allows for seasonals only in the volatility. Since these two constrained models are non-nested, a 2.5 percent significance level test is appropriate for each of these two tests, in order to ensure that the true size of these tests is 5 percent.
First, setting all the seasonals in mean returns equal to zero but retaining the seasonal scales yields:

\[ r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim S_{\alpha}(0, c_t). \]  \hspace{1cm} (3.4)

The LR test statistic for model (3.4) vs. model (3.3) equals 18.96. This test statistic is not significant at the 2.5 percent level, or even at the five percent level, using the critical value from the $\chi^2_{11}$ distribution. Thus, one is unable to reject the hypothesis of no seasonals in mean returns. Point estimates (not reported) would indicate that the biggest seasonals in mean returns occur in the months of September and November, these being -1.70 and 1.36 percent, respectively.\(^2\) Individual Wald tests would indicate significance of these two seasonals at the five percent level. However, these may well be spurious, given the nominal test size and the fact that there are eleven seasonals in all. A January effect, as documented in mean returns for smaller stocks in earlier studies, is not found in our real returns. Note that Seyhun (1993) also fails to find the January returns dominating returns in the other months for the equally-weighted portfolio of the largest decile of firms. Our failure to find the January effect in mean returns should not, therefore, necessarily be surprising, since value-weighted returns may be dominated by the behavior of large stocks.

Second, setting the scales in every month equal to a common value, but retaining seasonals in mean returns yields:

\[ r_t = \mu + \sum_{i=1}^{12} s_i I_{it} + \varepsilon_t, \quad \varepsilon_t \sim \text{iid } S_{\alpha}(0, c). \] \hspace{1cm} (3.5)

The LR test statistic for model (3.5) versus model (3.3) equals 23.10. The absence of volatility seasonals is rejected using the 2.5 percent critical value from a $\chi^2_{11}$ distribution. We may conclude from this pair of tests that seasonality is present in the volatilities but not in the means.

\(^2\) In Appendix C we report a positive seasonal in October and negative seasonals in March and April in the CPI-based inflation rate. Thus, the apparent seasonals in mean real returns for September and November do not appear to be merely an artifact of deflating nominal returns by the inflation rate.
The estimation results from model (3.4) are presented in Table E.2. A test of model (3.4) versus the completely non-seasonal model (3.2), reported in Table E.3, yields an LR test statistic of 23.26, which is also rejected at the 2.5 percent level. This test thus reinforces that one can reject the hypothesis of homoskedastic real returns in favor of seasonally heteroskedastic returns.

Individual Wald statistics indicate the strongest seasonals in the months of January and March. We, therefore, estimate a model with different scales for each of these months, but common scales for the remaining months (model V1). The LR test in Table E.3 indicates that we cannot reject the hypothesis that only these scales are different versus that all scales are different, model V1 vs. model (3.4). Moreover, we can reject the hypothesis that the scales in these two months are equal to the common scale for the remaining months, model (3.2) vs. model V1. We investigate further by setting each of the scales for these aberrant months in turn equal to the common scale for the remaining months. The LR tests in Table E.3 indicate that we can reject the hypotheses that either January or March have the same scale as the rest of the months (models V2 and V3 vs. model V1, respectively). Thus, model V1 best accounts for volatility differences across months.

In model V1, the January scale is 3.65, which is greater than the common scale of 2.64 for the other months. The scale for March is estimated to be 1.89. Thus, the January returns are, on average, more volatile by a factor of 1.38 than the returns in the rest of the year, whereas the returns in March are less volatile by a factor of 0.72. Thus, for the monthly post-1953 real value-weighted CRSP returns with dividends, the only January effect that is present seems to be manifested in the scales rather than the means of stock returns. This result is surprising, since Seyhun (1993) fails to find a January effect in the standard deviations of equally-weighted size-based decile monthly portfolio returns.

We have no a priori reason to expect volatility seasonals in January or March, but remove them because they do appear to be strongly significant, and because such volatility seasonals could be the source of some of the leptokurtosis in stock returns. The
fact that these are significant even when the disturbances are robustly modeled as stable indicates that they are not just fingerprinting a few unusual draws.

3.4 Persistence in volatility

One further stylized fact about stock returns that emerges from the literature is that the autocorrelation function of return volatility is positive, with a slowly decaying tail. Time-varying volatilities of daily returns (Nelson (1991), Danielsson (1994)), weekly returns (Hamilton and Susmel (1994)), and monthly returns in the pre-Depression period (Pagan and Schwert (1990)) have been modeled in the literature using a variety of techniques, including GARCH/EGARCH, Markov switching models, and stochastic volatility models. These studies find statistically significant time-varying volatility in returns.

However, other research, such as Diebold and Lopez (1995), reports weaker evidence on conditional heteroskedasticity, especially for the returns in the later years. Ghose and Kroner (1995) test for persistence in second moments in a number of financial time series, including daily logged differences of the SP500 index, using the Lagrange Multiplier (LM) test. Their LM test cannot reject homoskedasticity in the SP500 index at 5 percent critical values, although homoskedasticity is rejected in all the other series that they examine.

To investigate whether persistence in volatilities is significant in monthly real returns, we begin by extending model V1 with a GARCH(1,1) process driving the conditional scale, $c_t$:

$$r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim S_d(0, C_s c_t)$$

$$c_t^* = \omega + \beta c_{t-1}^* + \delta(r_{t-1} - \mu)/C_s$$

We restrict $\omega > 0$, $\beta \geq 0$ and $\delta \geq 0$. The term $C_s$ is designed to capture seasonality in scales, and is equal to one for all months, except January and March for which it is $C_{Jan}$ and $C_{Mar}$, respectively, to be re-estimated. When the errors are normal, and $C_s$ identically equals one, this model reduces to the familiar GARCH-normal process. For simplicity, we select the GARCH(1,1) specification above. This has by far been the most
popular specification used to describe stock return volatility. A non-seasonal GARCH-stable model similar to (3.6b) has been estimated for daily foreign currency returns by Liu and Brorsen (1995). The results of estimating (3.6) are reported in Table E.4. The table also reports the estimation results of a similar process for a non-seasonal stable GARCH model, G. The likelihood ratio tests for the null hypothesis of no GARCH (test for $\beta = \delta = 0$) are reported in Table E.5. The main results can be summarized as follows: there is a highly significant GARCH effect in monthly returns in our sample period, model V1 vs. model (3.6). Ignoring the seasonals in volatility leads to a similar conclusion, model (3.2) vs. model G. A GARCH(1,1) process captures the heteroskedasticity in returns better than our simple model with constant but seasonal scales (compare the maximized log-likelihood values for models G and V1). Interestingly, the volatility seasonals remain statistically significant, even after accounting for GARCH-like heteroskedasticity in the

---

3 Pagan and Schwert (1990) fit a GARCH(1,2) model with normal shocks for monthly returns from 1834-1925, while French, Schwert and Stambaugh (1987) fit a similar model to monthly returns from 1928-1984. Both these studies find only weak effects of the second MA term.

4 De Vries (1991) shows that stable subordinated processes with conditional scaling (SSCS) exhibit GARCH-like conditional heteroskedasticity, and, at the same time, imply stable marginal distributions. Thus, under certain restrictions, stable and GARCH-like processes are observationally equivalent from the viewpoint of the unconditional distribution. However, the volatility function of de Vries that generates the clustering phenomenon and implies a subordinated process with stable marginals involves an extra state variable. This kind of volatility function creates problems in estimating models of persistence that we set up later since, as we shall see, state-space models with multiple elements in the state-vector are not easily amenable to estimation procedures employed in this paper. We, therefore, abstract from the volatility specification of de Vries and, instead, adopt the usual GARCH formulation for conditional heteroskedasticity, with a modification to account for the volatility seasonals.

5 Lumsdaine (1996) shows that the effect of initial values in the GARCH volatility process on the properties of the parameter estimators in GARCH(1,1) and IGARCH(1,1) models is asymptotically negligible. Diebold and Lopez (1995) suggest setting the initial conditional variance (equal to $2c_0^2$, when it exists) equal to the sample variance at the first iteration and at subsequent iterations to the sample variance from a simulated realization with the estimated parameters (from the previous iteration). Here, we follow Engle and Bollerslev (1986), and initialize the GARCH process using estimates of $c_0$ based on sample values.
series, model G vs. model (3.6). In fact, the test statistic, $2\Delta \log L$, is now 18.15 as against 13.92 for the non-GARCH models (3.2) vs. V1.

GARCH-normal models for stock returns were originally motivated by the temporal clustering of big shocks in the empirical distributions of the daily and weekly returns. However, the empirical success of these models is partly due to their ability to generate fat tails, with finite or infinite unconditional variances. Due to this property, GARCH processes are often regarded in the literature as alternatives to iid stable processes. If GARCH-normal processes, indeed, adequately account for all the observed leptokurtosis in real returns, one would expect the evidence against normality to be eliminated, once GARCH effects are taken into account.

To investigate this issue, we estimate models (3.2), V1, G and (3.6) with normal shocks, and conduct likelihood ratio tests for normality (test for $\alpha = 2$) with each of these models. The results indicate that, as we account for heteroskedasticity better (going from models (3.2) to V1 to G to (3.6)), the estimated value of $\alpha$ rises, reflecting more nearly normal behavior of the shocks. Ignoring conditional heteroskedasticity, thus, seems to result in underestimation of the characteristic exponent. This downward bias has also been noted in the estimation of foreign currency returns by Liu and Brorsen (1995). However, the likelihood ratio statistics for $\alpha = 2$, reported in Table E.6, do not decline monotonically as we account for heteroskedasticity better. On the contrary, this test statistic actually increases from model (3.2) to (3.6) ($2\Delta \log L$ equals 30.27 for testing $\alpha = 2$ with model (3.2) as against 36.16 with model (3.6)). Non-normality remains highly statistically significant throughout. Thus, there is strong evidence that, even after accounting for the seasonal scales and GARCH-like behavior, real returns are still significantly non-normal.\(^6\)

\(^6\) A number of other studies find that GARCH-normal models do not adequately account for all the leptokurtosis in stock returns (see Hamilton and Susmel (1994), Diebold and Lopez (1995)). Many of these studies, consequently, fit a GARCH model, with the innovations drawn from an alternative leptokurtic distribution, such as the Student-$t$ distribution. Comparing these models goes beyond the scope of the present paper.
The estimated scales from model (3.6) are tabulated in Table E.7. These scales are plotted in Figure E.2, panel A. For comparison, we also plot the estimated scales obtained from model (3.6), when shocks are assumed normal, in panel B. Both figures reveal very similar and highly non-constant scales, with the scales from the stable model being somewhat smaller in general. Seasonality is clearly evident in both figures.

### 3.5 Persistence in mean returns

Visual inspection of the real return series (in Figure E.1) suggests runs of above and below average returns. In order to detect any predictable variation in stock returns, we set up an unobserved components model for real returns with seasonal and GARCH effects:

\[
\begin{align*}
    r_t &= x_t + \varepsilon_t, & \varepsilon_t &\sim S_\alpha(0, C_\varepsilon \varepsilon_t) \\
    (x_t - \mu) &= \phi(x_{t-1} - \mu) + \eta_t, & \eta_t &\sim S_\alpha(0, C_\eta \eta_t) \\
    c_t^n &= \omega + \beta c_{t-1}^n + \delta(r_{t-1} - E(r_{t-1}|r_1, r_2, \ldots, r_{t-2})) / C_s |^{n}.
\end{align*}
\]  

(3.7a, 3.7b, 3.7c)

Here, \( r_t \) is the observed one-period real return, \( x_t \) is an unobserved persistent component in the series, and \( \varepsilon_t \) and \( \eta_t \) are serially and mutually independent white noise processes. Any predictable variation in real returns is due to the presence of the persistent component \( x_t \), assumed here to follow a simple AR(1) process.\footnote{An unobserved components mean-reverting model for stock prices due to Summers (1986) has the form:}

\[
\begin{align*}
    p_t &= q_t + z_t \\
    q_t &= \mu + q_{t-1} + u_t \\
    z_t &= \phi z_{t-1} + v_t
\end{align*}
\]

where \( p_t \) is the log of stock price index, \( q_t \) is an unobserved random walk component, \( z_t \) is an unobserved stationary component (\( 0 \leq \phi < 1 \)), and \( u_t \) and \( v_t \) are serially and mutually independent white noise processes.

In terms of stock returns, this has the same structure as (3.7) where \( x_t = \mu + (\phi - 1)z_{t-1} \), \( \varepsilon_t = u_t + v_t \), and \( \eta_t = (\phi - 1)v_{t-1} \).
Brorsen (1995), an \( \alpha \) power is used in the GARCH-stable specification of (4.1c). The null hypothesis is that returns are random, apart from a non-zero mean, volatility seasonals, and volatility persistence, as described by model (3.6).\(^8\)

If the predictable component in (3.7) is significant, then \( E(r_t | r_{t-1}, \ldots, r_{t-11}) \) provides a useful forecast of returns. On the other hand, if \( c_\eta \) and/or \( \phi \) is negligible, then the returns are purely random, and may display only spurious periodicities.

The non-Gaussianity of the state-space model (3.7) creates complications in estimation, even without the presence of conditional heteroskedasticity. This is because the celebrated Kalman filter is no longer optimal due to the non-Gaussian nature of the shocks. However, the general recursive filtering algorithm due to Sorenson and Alspach (1971) provides the optimal filtering and predictive densities under any given distributions for the errors, and a formula for computing the likelihood function. Appendix B.1 gives these formulae. The recursive equations for computing the filtering and predictive densities are given in the form of integrals, whose analytical closed-form expressions are generally intractable, except in very special cases. In this paper, we

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8 Model (3.7) can detect both short- and medium-term memory in stock returns. However, it is incapable of capturing any “long-memory,” in the sense of an autocovariance function with a hyperbolic asymptotic rate of decay, that may be present. This is because the model has a reduced form ARMA representation and, hence, the dependence between successive realizations in a sequence generated by the model decays geometrically. “Long-memory” processes can be parametrically modeled by the ARFIMA class of models, with a fractional differencing parameter. The theory of ARFIMA models with infinite variance stable shocks can be found in Kokoszka and Taqqu (1995). Crato (1994) estimates ARFIMA models with Gaussian errors for monthly stock indexes of G-7 countries from 1950 through 1989, and fails to find any statistical evidence for long memory in all countries, except West Germany. His results support the conclusion reached by Lo (1991), who tests for long-memory in daily and monthly, value- and equally-weighted, CRSP returns using a modified rescaled adjusted range, Q or R/S, statistic. We abstract from the complications that long-memory models present in this paper, and presume (3.7) to adequately capture any persistence that there may exist in stock returns.
numerically evaluate these integrals. Details on the numerical implementation procedure adopted in this chapter are given in Appendix B.3.

The maximum likelihood estimation results are presented in the first column of Table E.8. The results indicate an estimated $\alpha$ of 1.892. The AR coefficient of the persistent component of returns is estimated to be 0.42. The estimated signal-noise scale ratio is about 0.10. A plot of the mean of the filter density, $E(x_i|r_1,r_2,\ldots,r_i)$, appears in Figure E.3, along with the observed real returns. The figure shows a predictable component that is practically constant, indicating that any variation in it is of little importance in forecasting real returns. For comparison, we reproduce the estimation results of the constant mean model (3.6) from Table E.4 in the second column of Table E.8.

A statistical test for no persistence in returns can be formulated as a test of the null hypothesis, $\phi = 0$. In this case, the two shocks, $\varepsilon_t$ and $\eta_t$, are not separately identified, so we may add $c_\eta = 0$. It is therefore not clear whether to use one or two degrees of freedom when testing for the null hypothesis with a standard likelihood ratio test. The likelihood ratio statistic is found to be 3.42. We cannot reject zero persistence in mean at the five percent level, using critical values from the $\chi^2_1$ or the $\chi^2_2$ distribution.

To determine what the inferences on predictability would be under the assumption of normality, we also estimate model (3.7), setting $\alpha = 2$. In this instance, we use the Kalman filter for estimation as it is not only optimal when $\alpha = 2$, but also computationally exact and more expedient. These estimation results, also reported in

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9 The derivation of the asymptotic $\chi^2$ distribution of the likelihood ratio statistic requires that the likelihood function be approximately quadratic in the region in which the null hypothesis and the global optima lie. If a particular parameter is not identified under the null, then the likelihood function is flat in the direction of that parameter. Therefore, the standard likelihood ratio test is inapplicable. Hansen (1992) derives a bound for the asymptotic distribution of a standardized likelihood ratio test statistic that is applicable in such situations. Hansen notes that, since his theory only provides a bound for the asymptotic distribution (as against the actual asymptotic distribution itself), use of his test may result in underrejection of the null and a subsequent loss of power. Furthermore, the implementation of his test is computationally very intensive in general, and more so for our particular problem. Therefore, we refrain from using his test here. Monte Carlo simulations are again not attempted for the same reason.
Table E.8 (columns 3 and 4), similarly indicate a sizeable estimate of $\phi$, but a small signal-noise scale ratio. The standard LR test fails to reject $\phi = 0$ and $c_{\eta} = 0$ using $\chi^2_1$ or $\chi^2_2$ critical values at even the ten percent level.

3.6 Conclusions

Our principal findings are as follows. The monthly CRSP value-weighted real stock returns for the post-1952 period display highly significant leptokurtosis. The estimated value of the characteristic exponent, $\alpha$, is about 1.89, well away from the value of 2 characterizing normal behavior. Stock returns exhibit significant persistence in volatility that can be well-described by a GARCH-like process. However, a GARCH process, with normal innovations, cannot fully account for the observed leptokurtosis in the series, as claimed in some prior studies. Even the conditional densities are leptokurtic enough to warrant using stable shocks. Although we do not find evidence of any seasonals in mean returns, the months of January and March exhibit statistically significant seasonals in volatility.

Our results concerning the predictability of monthly stock returns are negative. Our state-space models fail to reveal a statistically significant persistent signal in returns, after taking into account the non-normality, seasonality, and volatility persistence in the series. The efficiently estimated expected real return is 0.764 percent per month (9.168 percent per annum), substantially more than the 0.577 percent per month (6.924 percent per annum) obtained with the inefficient conditionally normal model (3.6). Although real stock prices do not take an iid random walk, they do appear to follow a heteroskedastic martingale.

There are several ways in which our models can be further refined to more closely reflect the empirical distributions of stock returns. First, our methodology treats stock return expansions and contractions symmetrically, whereas Cochran and Defina (1995) report asymmetries in return dynamics. Models that are non-linear in the specification of the conditional mean, therefore, could be fruitful.
Second, our specification for conditional heteroskedasticity does not take into account any asymmetric response of stock volatility to positive and negative return shocks. Such a ‘leverage effect’ has been reported by a number of authors (Nelson (1991), Hamilton and Susmel (1994)). Asymmetry in the conditional stable distribution itself is another aspect worth exploring in future research.

Multivariate models have also been used to describe stock return volatility.\textsuperscript{10} While efficiency gains may accrue from use of multivariate models, these present computational difficulties due to multiple elements in the state-vector that are hard to overcome with the estimation techniques adopted in this paper. However, Monte Carlo integration techniques may ultimately overcome these difficulties.

\textsuperscript{10} Hamilton and Lin (1994) set up a bivariate model for stock volatility and the growth rate of industrial production, in which the volatility of stock returns, modeled as an ARCH-L process (ARCH with leverage effect), and the mean growth rate of industrial production are subject to occasional shifts that are governed by a common Markov process. They find that this bivariate model forecasts stock volatility better than a univariate switching ARCH-L model.
ON MODELING REAL GNP: NON-NORMALITY AND NON-LINEARITY, BUT NO LONG MEMORY

Summary: We examine quarterly U.S. real GNP data for evidence of non-linearity and long memory. Since the statistical evidence on non-linearities in the conditional mean could be influenced by the presence of outliers, or by a failure to model changing volatilities, we explicitly account for outliers by assuming that the innovations are drawn from the stable family, and model time-varying volatility by a GARCH process. Our results indicate statistically significant non-linearities in the conditional mean. Evidence on long memory is weak, and is sensitive to the specification of the short-run dynamics. Linear Gaussian models, and non-linear conditionally heteroskedastic models of the GARCH class, seem inadequate for understanding the dynamics of this time series.

4.1 Introduction

The study of income dynamics is a crucial component in understanding economic fluctuations, and has deservedly received an enormous amount of research attention. While most studies using parametric techniques have typically focused on Gaussian
models, the conditional distributions implied by these models may not accurately portray the sample path of this important time series. For instance, Blanchard and Watson (1986) investigate whether fluctuations in economic activity are caused by an accumulation of individually unimportant small shocks, or by infrequent large shocks, and conclude that the economy is instead characterized by a mixture of large and small shocks. Balke and Fomby (1994) report evidence of excess kurtosis in several macroeconomic time series, including quarterly real GNP, and attribute this to the presence of outliers. These studies, therefore, argue against modeling these series with Gaussian errors on account of their thin tails.

The analysis of Balke and Fomby (1994) also indicates a clustering of outliers across time, which is symptomatic of a persistent volatility process. This argues against homoskedastic models for real GNP in favor of conditionally heteroskedastic models, such as the ARCH or GARCH models (see also French and Sichel (1993)). While it is well-known that these models have leptokurtic unconditional distributions and may, therefore, account for some of the outliers detected by Balke and Fomby (1994), it remains to be investigated whether GARCH-normal models can account for all the observed leptokurtosis in the data. It is conceivable that real GNP is better described by a conditionally heteroskedastic model driven by leptokurtic shocks.

Of late, evidence has been accumulating in the literature regarding possible non-linearities in business cycles. The non-parametric methods of Neftci (1984) and Brunner (1992) indicate that expansions are long periods of slow growth rates, but recessions are sharp and short-lived. While these non-linearities could arise from asymmetric impulses being propagated by a linear economic structure, the parametric methods of Beaudry and

11 As a matter of practical concern, although it may be difficult to distinguish between true small shocks and a large shock arising from a succession of small shocks due to the choice of the sampling interval, Blanchard maintains that the forecast decompositions of GNP in their study do not seem to indicate that this is indeed the case.

12 Indeed, in the finance literature, it is now well established that GARCH normal models cannot account for the all observed leptokurtosis in stock returns, and these are, therefore, frequently modeled using GARCH models with thick-tailed errors, such as the Student-\(t\) or stable distributions. Recently, Baillie et al. (1996) also fit a GARCH model with Student-\(t\) innovations to the CPI inflation of ten different countries, including the U.S.
Koop (1993) and Potter (1995) indicate that the asymmetries are in fact driven by a non-linear propagation mechanism. If the world is indeed non-linear with or without asymmetric impulses, as these studies seem to suggest, then business cycle theorists need to look at more than the covariance properties in order to assess the validity of their models. Measures of long-run persistence of shocks to real income, reported in earlier studies based on linear models, are likely to be biased.

However, the evidence on non-linearities in the literature is somewhat controversial. For instance, while Falk (1986) fails to replicate Neftci’s (1984) results in other macro series, Sichel (1989) attributes Neftci’s findings to an error in his computations. DeLong and Summers (1986) and Diebold and Rudebusch (1990) find only weak evidence of asymmetries. Balke and Fomby (1994) report weakened evidence of non-linearities once outliers are taken into account, a point also echoed by Scheinkman and LeBaron (1989). Failure to account for conditional heteroskedasticity could also induce spurious rejection of linearity, as also demonstrated by Scheinkman and LeBaron.

The purpose of this paper is to investigate whether non-linearities exist in business cycles, taking into account the non-normality and conditional heteroskedasticity in the series. While the study by Balke and Fomby (1994) relies on outlier identification techniques and intervention analysis, we explicitly account for the aberrant observations by modeling them within the framework of thick-tailed distributions. We use symmetric stable shocks, as these are natural generalizations of Gaussian distributions and possess central limit attributes.

Another issue that we investigate is long memory in real income. Long memory models have recently been used to describe real GNP by Sowell (1992a) and inflation by Baillie et al. (1996). While Sowell considers several parameterisations of homoskedastic long memory models with normal errors, Baillie et al. fit long memory conditionally heteroskedastic models (GARCH), with non-normal innovations. It would be interesting to extend Sowell’s work, by incorporating asymmetries and conditional heteroskedasticity, and see how the results compare when shocks are also assumed to be non-normal.
We use seasonally adjusted quarterly U.S. real GNP data, measured in constant 1987 dollars and spanning 1947.I through 1995.I, from Citibase for analysis. Growth rates are defined as 100 times the first differences of natural logarithms of real GNP. Figure F.1 plots the level and the growth rate of real GNP during this period.

We first consider non-normality of the series with a simple homoskedastic model using stable shocks in section 2, and then extend by incorporating conditional heteroskedasticity in section 3. We entertain nonlinearities in the specification of the conditional mean in section 4, and investigate fractional differencing in section 5. Some concluding remarks are offered in the last section.

### 4.2 A Simple Homoskedastic Model

Sample autocorrelations for the growth rates of quarterly U.S. real GNP equal 0.38 and 0.23 at the first and second lags, while the sample partial autocorrelation at the second lag equals 0.10. This suggests modeling the growth rate as an AR(2) or an MA(2) process. Both models have been fit to the series in the past. We begin with a simple AR(2) model for the growth rates, $\Delta y_t$:

$$\begin{align*}
(1 - \phi_1 L - \phi_2 L^2)(\Delta y_t - \mu) &= \varepsilon_t, \\
\varepsilon_t &\sim \text{iid } N(0, \sigma^2)
\end{align*} \tag{4.1}$$

Here, $\mu$ is the mean growth rate, $L$ is the lag operator, and $\Delta$ is the first-difference operator, $(1 - L)$. Conditional maximum likelihood (ML) yields the estimates tabulated in Table F.1. The first order autoregressive parameter $\phi_1$ is estimated to be 0.345 and significantly different from zero at the 95 percent significance level, whereas $\phi_2$ is estimated to be 0.096 and statistically insignificant. The mean growth rate is estimated to be 0.772 percent per quarter, or 3.090 percent annually.

Relaxing the assumption of normality in favor of the more general stable distributions results in the following model:

$$\begin{align*}
(1 - \phi_1 L - \phi_2 L^2)(\Delta y_t - \mu) &= \varepsilon_t, \\
\varepsilon_t &\sim \text{iid } S_{\alpha}(0, c)
\end{align*} \tag{4.2}$$

where $S_{\alpha}(0, c)$ represents a symmetric stable distribution with characteristic exponent $\alpha$, location 0, and scale $c$. Conditional ML estimates of the parameters in this model are
reported in the second column of Table F.1. The tail parameter $\alpha$ is estimated to be 1.817, well away from the value of 2 characterizing normal shocks. The estimates of the other parameters are similar to those from the normal model. The maximized log-likelihood shows an increase from -253.49 to -250.44. A test for $\alpha = 2$, which is a formal test for normality, can be based on the likelihood ratio (LR) statistic. However, the distribution of this statistic is non-standard, since the null hypothesis lies on the boundary of admissible values for $\alpha$, and, hence, the standard regularity conditions are not satisfied. Based on the small sample Monte Carlo critical values tabulated for a simple location model by McCulloch (1996a), we can easily reject normality in favor of $\alpha < 2$ at the 0.01 level of significance. In order to test for residual serial correlation in our model, we cannot directly apply the Box-Pierce test to the stable residuals since these violate the finite second moment assumptions required for deriving the asymptotic distribution of the test statistic. However, the Box-Pierce test, when applied to the normal residuals, failed to reveal any significant serial correlation, which we take as suggestive of no misspecification in the stable model either. Although $\phi_2$ is estimated to be statistically insignificant, we nonetheless retain it for comparison of our results with earlier studies.

4.3 Conditional Heteroskedasticity

Whereas the disturbances in the above model are assumed to be iid there is some evidence of conditional heteroskedasticity of the GARCH/EGARCH form in real income (French and Sichel (1993), Balke and Fomby (1994)). Incorporating first-order GARCH effects in $\varepsilon_t$ above, our model for GNP becomes:

\[
(1 - \phi_1 L - \phi_2 L^2)(\Delta y_t - \mu) = \varepsilon_t, \quad \varepsilon_t \sim S_\alpha(0, c_t) \tag{4.3a}
\]

\[
c_t^\alpha = b_1 + b_2 \varepsilon_{t-1}^{\alpha} + b_3 |\varepsilon_{t-1}|^\alpha \tag{4.3b}
\]

Equation (4.3b) describes the evolution of the scale of the conditional distribution, which reduces to the familiar GARCH(1,1) process for the conditional variance when $\alpha = 2$, as is the case when shocks are normal.
Estimation of the above model was carried out by conditional ML, treating \(c_0\) as a fixed but unknown quantity to be estimated with the rest of the parameters and \(\varepsilon_0 = 0\).\(^{13}\) Results are presented in Table F.2 for the stable and normal models. The maximized log-likelihood showed an increase of 8.29 over the homoskedastic model in the stable case. Lumsdaine (1996) derives the asymptotic normality of the ML estimator in GARCH(1,1) and IGARCH(1,1) models, and Lumsdaine (1995) studies some finite sample properties of these estimators and related test statistics in these models. A test for homoskedasticity can be based on the LR, although \(b_1\) and \(c_0\) are not independently identified under the null hypothesis (one can be deduced as a trivial transform of the other). However, we are easily able to reject homoskedasticity at better than the 0.005 level using the critical values from a \(\chi^2\) distribution with either 2 or 3 degrees of freedom. This shows that homoskedasticity is strongly rejected, even after accounting for outliers. This contrasts with the analysis of Balke and Fomby (1994), who report weakened evidence against homoskedasticity versus GARCH alternatives, once outliers are taken into account.

The GARCH persistence parameter \(b_2\) is estimated to be 0.801, and \(b_3\) 0.082, with their sum well below unity. So the stationary GARCH formulation seems adequate, with the parameter estimates giving no great concern for considering an integrated GARCH (IGARCH) process.\(^{14}\) With respect to the estimates of the other parameters in

\(^{13}\) Lumsdaine (1996) shows that the effect of the initial values on the properties of the parameter estimators in GARCH(1,1) and IGARCH(1,1) models is asymptotically negligible. Engle and Bollerslev (1986) suggest using initial values based on the sample estimates. Diebold and Lopez (1995) suggest setting the initial conditional variance (equal to \(2c_0^2\), when it exists) equal to the sample variance at the first iteration and at subsequent iterations to the sample variance from a simulated realization with the estimated parameters (from the previous iteration). As our parameter estimates (to be discussed shortly) indicate, \(b_3\) is estimated to be quite small. However, \(b_2\) is by no means negligible. A poor choice of \(c_0\) can introduce non-trivial transients into the volatility process and affect the point estimates of \(b_2\). Therefore, it is prudent to estimate the starting value.

\(^{14}\) Although the hypothesis of an integrated GARCH process could be formally tested (for instance, by testing for \(b_2 + b_3 = 1\) with, say, an LR test (see Lumsdaine (1995)), the statistical
the model, the introduction of conditional heteroskedasticity leads to an increase in the point estimate of $\phi_2$, which jumps up to 0.117 from its previous estimate of 0.055. The LR test statistic for normality equals 3.68, which is rejected at the 0.02 level. Thus, although the LR statistic declines with the introduction of GARCH, non-normality is still quite strong. This is also evidenced by the low (as compared to $2^{1/2}$) $\alpha$ estimate of 1.794.

The statistically insignificant autoregressive coefficient $\phi_2$ implies a first-order autoregressive system for the growth rates. A deterministic first-order difference equation does not admit cyclic solutions. Even if we ignore the statistical insignificance of $\phi_2$, a positive point estimate for it rules out solutions to the deterministic version of (4.3) that imply periodic cycles.\(^{15}\) Furthermore, one could construct confidence ellipses around $\phi_1$ and $\phi_2$, and test formally whether cyclic solutions are feasible at all for some combinations of these autoregressive parameters, as in McCulloch (1975). However, it seems unlikely that we will find significant evidence of periodicity in this time series that can be exploited for forecasting turning points in economic activity from its observed history. These results echo the sentiments in McCulloch (1975), who concludes that business ‘cycles’ are indistinguishable from Monte Carlo cycles that exist only in the eyes of the beholder.

4.4 Asymmetric expansions and contractions

Our simple symmetric AR(2) model in growth rates implies that impulse responses to negative and positive innovations are symmetrical. This means that negative and positive shocks of equal magnitude have the same-sized effect on future output at any horizon, that the persistence of positive and negative shocks is symmetrical, and that it takes just as long to get out of a recession as for a boom to end. Further symmetry is imposed on the model by assuming symmetrically distributed shocks. However, the idea evidence that we present concerning the main issues discussed in this paper is unlikely to be materially affected by whether the volatility process is explicitly modeled as persistent or not.

\(^{15}\) A standard result in solving difference equations implies that a deterministic second-order system admits periodic solutions if and only if $\phi_2 < -\phi_1^2 / 4$.  

53
that business cycles may not actually be symmetric is an old one (see Potter (1994) for a brief account of the origin and history of the business cycle asymmetry debate).

The presence of asymmetries essentially implies that either the innovations to real income are asymmetric but the impulse transmission mechanism linear, or that innovations are symmetric but the transmission mechanism is nonlinear, or that innovations are asymmetric and the transmission mechanism is also non-linear. Such evidence rules out linear Gaussian ARMA processes for representing real GNP. Models with asymmetric $\alpha$-stable impulses are harder to work with because there currently exists no fast computational technique for evaluating the probability density function of these random processes. Therefore, we shall restrict ourselves in this chapter to consideration of models with symmetric impulses.

Mittnik and Niu (1994) discuss several non-linear time series models in the context of business cycle asymmetries. It is useful to begin with a very general class of models, called the Single Index Generalized Multivariate Autoregressive (SIGMA) models, discussed in Potter (1995). Let $y_t$ represent the observed univariate time series, $z_t$ an unobserved time series, $H_t$ the single index, assumed to be a continuous map from $\{y_t, z_t\}$ to the real line, and $F(.)$ a function from the real line to the unit interval with at most a finite number of discontinuities. Then, a univariate first-order SIGMA model is defined as:

$$y_t = \alpha_1 + \alpha_2 F(H_t) + \{\phi_1 + \phi_2 F(H_t)\}y_{t-1} + \{\psi_1 + \psi_2 F(H_t)\}v_t$$

where $v_t$ are iid with mean zero and unit variances. This model nests the AR(1) model (homoskedastic and heteroskedastic), deterministic and stochastic time varying parameter models (see Tucci (1995) for a critical introduction), and many of the non-linear models that have been used to capture asymmetries, such as threshold autoregressions (TAR, see Tong and Lim (1980)) or the regime-switching model of Hamilton (1989).

TAR models are piecewise linear autoregressions, and can approximate a general nonlinear time series model of the form

$$y_t = f(y_{t-1}) + \varepsilon_t,$$
with \( f(.) \) being piecewise continuous, arbitrarily closely. They are also capable of capturing jump phenomena. Moreover, while linear models (without drift) have a constant steady-state, called the limit point, TAR models are capable of displaying limit cycles.\(^{16}\) The specification of the threshold variable, in terms of which the single index restriction is defined, is crucial as it plays a key role in determining the nonlinear nature of these models.\(^{17}\) Because TAR models exhibit discontinuity at the threshold, and are, therefore, considered somewhat counterintuitive, smooth transition autoregressive (STAR) models, with certain continuity constraints that permit modeling a continuum of states, have also been considered.\(^{18}\)

Testing the null hypothesis of linearity within the framework of threshold autoregressions gives rise to nuisance parameters. For instance, in Potter’s (1995) model of U.S. GNP (see footnote 17), the threshold and the delay parameters are not identified under the null hypothesis of a single regime. Standard distribution theory does not go through under these conditions. Hansen (1996) derives the asymptotic distribution of

\(^{16}\)By no means should this be construed as a drawback of these models, since equilibrium paths characterized by fluctuations, both periodic and chaotic (limit cycles with infinite period), can occur in optimizing models, even in the absence of stochastic exogenous shocks (see Boldrin and Woodford (1990) for a survey of these models displaying endogenous fluctuations and chaos).

\(^{17}\) When the restrictions are defined in terms of the observed series, \( y_t \), these are called self-exciting threshold autoregressive (SETAR) models. Potter (1995) estimates a single restriction version of this model for log real GNP. The restriction is defined in terms of whether \( \Delta y_{t-d} > r \), where \( y_t \) is log real GNP, \( d \) is a delay parameter and \( r \) the threshold parameter. This model is essentially a piecewise linear autoregression in which the model parameters (i.e. the intercept, the AR coefficients, and the error variance) assume different values, depending on whether the single index restrictions are satisfied or not. Such a switching regression model, with the restriction defined in terms of the current depth of recession (defined in model (4.4) subsequently) is estimated by Beaudry and Koop (1993).

\(^{18}\) Terasvirta and Anderson (1992) use two different specifications: \( H_1 = -\gamma (y_{t-d} - r), \gamma > 0 \) with the logistic function for \( F(.) \), termed the logistic STAR (LSTAR) model, and \( H_2 = -\gamma (y_{t-d} - r^2), \gamma > 0 \) with \( F(.) = 1 - \exp(.) \), termed the exponential STAR (ESTAR) model. These models, estimated for the industrial production data of several OECD countries, indicate that business cycles, especially for the United States, are characterized by strong and swift recoveries from deep recessions, with booms showing more gradual decay.
standard test statistics in these situations, which are found to depend on the nuisance parameters themselves. However, a transformation of these statistics leads to a uniform distribution on the unit interval. On application of his test to Potter’s model, Hansen is unable to reject the hypothesis that the threshold non-linearity, detected by Potter in U.S. real GNP, is caused by sampling variation.

An alternative non-linear formulation that Beaudry and Koop (1993) use is obtained by augmenting the ARMA model for the growth rates with an asymmetry term. Their model for real GNP is:

\[ \Phi(L)\Delta y_t = \mu + \{\Omega(L) - 1\} \text{CDR}_t + \Theta(L)\varepsilon_t, \]

where CDR is defined as the gap between the current level of output and the economy’s historical maximum level, i.e. \( \text{CDR}_t = \max\{y_{t-j}\}_{j=0}^\infty - y_t \), \( \Phi(.)\), \( \Theta(.) \) and \( \Omega(.) \) are lag polynomials of orders \( p \), \( q \) and \( r \), respectively, with \( \Phi(0) = \Theta(0) = \Omega(0) = 1 \). Although addition of the non-linear term into a standard ARMA framework is ad hoc, this model has the virtues of simplicity and parsimony. It nests the standard ARMA models, and does not give rise to any nuisance parameters when testing for the significance of the non-linearities. Therefore, standard testing procedures are applicable. Moreover, it permits recessions to be less or more persistent than expansions, depending on the parameter estimates. For instance, consider a specific parameterisation of this model, where \( \Phi(.) \) and \( \Theta(.) \) are of order zero, and \( \Omega(.) \) is order one. Then, a positive \( \omega_1 \) implies that negative shocks are less persistent whereas a negative \( \omega_1 \) implies the opposite. \( \omega_1 = 0 \) reduces to a random walk model with drift. On estimation of (4.4) for different values of the lag orders, \((p,q,r)\), the authors find the \((2,0,1)\) model to be the preferred parameterisation.

Incorporating the above first-order asymmetry term into our AR(2) model yields:

\[
(1 - \phi_1 L - \phi_2 L^2)(\Delta y_t - \mu) = \omega . \text{CDR}_{t-1} + \varepsilon_t, \varepsilon_t \sim S_{\omega}(0, \sigma^2).
\]

\[
\varepsilon_t^\alpha = b_1 + b_2 \varepsilon_{t-1} + b_3 |\varepsilon_{t-1}|^\alpha
\]

Beaudry and Koop (1993) estimate their model under the assumption of iid normal distribution for \( \varepsilon_t \). Our model extends by incorporating conditional heteroskedasticity
and assuming the more general stable distribution. It nests our symmetric model (4.2). The presence of the lagged CDR term makes this model a non-nested version of the SIGMA models.

A plot of the CDR variable for the U.S. appears in Figure F.2. Estimation of (4.5) by conditional ML yields the estimates reported in Table F.3. For the stable model \( \omega \) is estimated to be 0.257, and significantly different from zero by the LR test at better than the 0.025 level. This indicates that the rejection of linearity, reported in earlier studies, is not driven by the presence of outliers (see Tsay (1988) for an illustration of this possibility). In fact, accounting for outliers seems to increase the degree of asymmetry and the significance of the test. This result is in sharp contrast to the conclusions of Balke and Fomby (1994) and Scheinkman and LeBaron (1989), whose tests for nonlinearity weaken once outliers are taken into account.

Failure to account for changing volatility could spuriously induce rejection of linearity (see Scheinkman and LeBaron (1989) for an illustration with GNP data). Our results indicate that the findings of Beaudry and Koop (1993) were not driven by this omission. However, accounting for conditional heteroskedasticity does lower the degree of asymmetry (Beaudry and Koop report an \( \omega \) estimate of 0.380 for their homoskedastic normal model). Curiously, inclusion of the asymmetry term leads to a substantial increase in the estimate of \( \phi_2 \), which is also now statistically significantly different from zero. The estimate of \( \alpha \) is slightly higher at 1.817, although the LR test indicates stronger rejection of normality (LR statistic equals 5.37, 1 percent critical value equals 4.76).

The subject of persistence in real GNP has received intense academic interest. One commonly used measure of persistence is the revision in a long-run forecast following a one-period shock, where the revision is measured in percentage terms relative to the initial shock. For linear models, this corresponds to the sum of the coefficients in the implied moving average representation of the process, and is independent of both the size of the shock and the history of the series. Studies report widely differing estimates for U.S. real GNP. For instance, Box-Jenkins ARIMA models typically yield estimates over 1.6, whereas unobserved components models often result in estimates below 0.7.
Our homoskedastic linear stable model (4.2) implies an estimate of about 1.83, which agrees with the estimates reported in Watson (1986) and Clark (1987) using linear Gaussian models.

For asymmetric models, persistence depends on both the size of the innovations and the history of the series. To understand how this varies with the innovations, we plot in Figure F.3 the persistence factor at a ten year horizon, for various sizes of the initial shock. The persistence factor is evaluated relative to a steady-state base case where the past shocks have been identically set to zero. Figure F.3 shows that all positive shocks lead to a persistence factor of about 2.12 with our conditionally heteroskedastic non-linear model (4.5). The effect of negative shocks is generally lower, and decays to zero as the size of the shock increases. Thus, ignoring asymmetries leads to underestimation of the persistence of positive shocks and overestimation of the persistence of negative shocks.

Linear models exhibit impulse responses that are simply proportional to the size of the shock and independent of the history of the series to date. But, for asymmetric models, these, like the persistence measures, are sensitive to both the size of the shock and the history of the series. In Figure F.4 we plot the impulse responses to a negative and a positive innovation of 1.5 percent occurring in period 1, these being once again constructed relative to a base case where past innovations are identically zero. Point estimates obtained from model (4.5) indicate that a positive innovation leads to a 3.2 percent permanent rise in output whereas an equally-sized negative innovation leads to only a 2.3 percent downward revision. The permanent effect of a 1.5 percent negative growth shock on real income is in sharp contrast to Beaudry and Koop’s (1993) finding of a hump-shaped response of real GNP to recessionary shocks, with the effect of recessions being completely reversed after just 8-12 quarters. Our estimated model displays this behavior only at somewhat higher values of the shock size.\footnote{Standard errors for the persistence factor and the impulse responses, plotted in Figures F.3 and F.4, can be computed based on their asymptotic distributions, as in Beaudry and Koop (1993). However, Koop (1996) shows that the small-sample distributions of these estimators are biased, skewed, fat-tailed, and multi-modal. He cautions that standard errors based on their asymptotic}

19 Ignoring non-
normality, or conditional heteroskedasticity, or both, does not seem to greatly alter the measure of persistence or the behavior of the impulse responses.

4.5 Long memory

Our models thus far have been predicated on the assumption of a unit root in real GNP. While this assumption has been the focus of much recent debate in the unit root literature, viewing time series as either unit-root nonstationary or trend-stationary is actually very restrictive. For, if one is willing to admit a continuum of possible values for the differencing parameter, the unit root and trend stationary alternatives emerge as very special cases, perhaps undeserving of all the recent research attention. The resulting fractionally integrated models are capable of exhibiting persistence, evident in the slow hyperbolic decay of their autocorrelation functions (see Granger and Joyeux (1980), and Hosking (1981) for an introduction to fractional differencing, and Baillie (1996) for an exhaustive survey). This is in contrast to the short-memory ARMA models, whose autocorrelations decay at the much faster geometric rate.

Since long memory ARFIMA models are frequently defined in terms of the rate of decay of their autocovariances, or equivalently, their autocorrelations\(^{20}\), their extension to infinite-variance stable shocks is not immediate. Kokoszka and Taqqu (1995) develop the theory of fractionally differenced ARMA time series with infinite variance stable innovations, establishing conditions for their existence and invertibility. They also study the asymptotic dependence structure of these processes in terms of alternate measures of dependence that are applicable to infinite variance stochastic processes. They show that, as in the finite variance cases, these measures asymptotically follow power functions, reflecting the long memory feature of these series.

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approximations can be very misleading in these circumstances. He estimates the exact finite-sample posterior density of these estimators by Bayesian methods. Alternatively, one could use the bootstrap, or the bias-adjusted bootstrap, as in Kilian (1996), to compute these standard errors. \(^{20}\) Two definitions are in vogue. One is based on the hyperbolic decay of the autocovariances, while the other is based on the spectral density being either zero or infinite at the zero frequency (see Baillie (1996)).
Diebold and Rudebusch (1989) estimate the fractional differencing parameter for quarterly U.S. real GNP to be about 0.9, and fail to reject the hypothesis of a unit root. However, Agiakloglou et al. (1993) show that the two-step estimation procedure, due to Geweke and Porter-Hudak (GPH), adopted by Diebold and Rudebusch is subject to considerable bias in many situations, leading to incorrect inferences regarding the long-run behavior of a time series. Sowell (1992a) demonstrates the extent of this bias in the estimates of Diebold and Rudebusch, and attributes it to contamination due to high-frequency components. His exact ML procedure yields an estimate of 0.41 for the differencing parameter. His tests, based on small-sample critical values generated by Monte Carlo simulations, reject the one-sided hypotheses that GNP is trend or difference stationary versus the fractional alternative at the 90 percent significance level, although a 95 percent test fails to reject these hypotheses.

Such studies argue against first differencing of the levels variables, as we have done for real GNP thus far. Allowing for fractional integration results in the following model for real income:

\[(1 - \phi_1 L - \phi_2 L^2)(1 - L)^d (\Delta y_t - \mu) = \omega \cdot CDR_{t-1} + \varepsilon_t, \varepsilon_t - S_{\varepsilon_t}(0, \varepsilon_t) \quad (4.6a)\]

\[c^a_t = b_1 + b_2 \varepsilon_{t-1}^a + b_3 |\varepsilon_{t-1}|^a \quad (4.6b)\]

where \(d\) is no longer restricted to integer values. Consider the case when \(\omega = 0\). Then, (4.6) reduces to the standard ARIMA model with integer differencing when \(d = 0\), i.e. the unit root case, whereas \(d = -1\) indicates overdifferencing or trend stationarity. In the

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21 For any real number \(d > -1\), the difference operator \((1 - L)^d\) is defined by means of the Binomial expansion, \(\sum_{j=0}^{\infty} \pi_j L^j\), where \(\pi_j = \frac{\Gamma(j - d)}{\Gamma(j + 1)\Gamma(-d)}\), and \(\Gamma(.)\) is the gamma function,

\[
\Gamma(x):= \begin{cases} 
\int_0^{\infty} t^{x-1} e^{-t} dt, & x > 0, \\
\infty, & x = 0, \\
x^{-1}\Gamma(1 + x), & x < 0.
\end{cases}
\]
case of Gaussian errors, the existence of a stationary causal and invertible solution to (4.6), with \( \omega \) restricted to zero, requires \(|d|<0.5\) (see Brockwell and Davis (1991)). For the \( \alpha \)-stable shocks, Kokoszka and Taqqu (1995) show that a unique causal MA(\( \infty \)) representation exists if \( \alpha(d-1)<-1 \). This implies that \( d \) can be positive only when \( \alpha>1 \). Further, for (4.6), with \( \omega=0 \), to be a solution to an AR(\( \infty \)) process, we need \( \alpha>1 \) and \(|d|<1-1/\alpha \). Consequently, we shall restrict \( \alpha \) and \( d \) in (4.6) to satisfy these constraints in the remainder of this section, in order to force our estimated models to possess causal and invertible representations.

The exact full-information ML method for estimating ARFIMA models due to Sowell (1992b) is applicable only when the errors are iid normal.22 However, Baillie et al. (1996) note that implementing Sowell’s full ML procedure for more complicated models, such as non-normal or conditionally heteroskedastic models or both, is likely to be either computationally extremely demanding or completely intractable. Instead, they use the conditional sum of squares (CSS) estimator, originally proposed in the context of ARFIMA processes by Hosking (1984), to estimate their ARFIMA-GARCH models, with normal or Student-\( t \) errors.

The CSS estimation of ARFIMA models consists of fitting an ARMA model to the series, \((1-L)^d(\Delta y_t - \mu)\), obtained by expanding the differencing operator, \((1-L)^d\), with the Binomial expansion and truncating the infinite series at the first available observation. The CSS estimator is discussed in the context of ARMA models by Box and Jenkins (1976). Its asymptotically normal distribution for the ARFIMA case, when the mean is known and the errors are normal, is derived in Li and McLeod (1986).23 The CSS

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22 The method is based on computing the autocovariance function of the observed series in terms of nonlinear transformations of hypergeometric functions. Chung (1994) provides alternate expressions for these autocovariances that can be easily and accurately evaluated by computers.

23 Recently, Chung (1996) has proposed estimating a two-parameter long memory process, the Gegenbauer process or the Generalized ARMA (GARMA) process, by the conditional sum of squares. This process generalizes the one-parameter long memory process that we have been discussing, and is characterized by hyperbolic and sinusoidal decay in autocorrelations. The simplest GARMA process is given by \((1-2\eta L + L^2)^d x_t = \varepsilon_t\), where \( \varepsilon_t \) is white noise, \(|\eta|\leq 1\).
procedure is asymptotically equivalent to full MLE. Some properties of the CSS estimator in the context of ARFIMA models, particularly with respect to its bias, are discussed by Baillie et al. (1996). They conclude that the CSS estimator performs quite satisfactorily in comparison to Sowell’s (1992b) exact MLE, while being computationally feasible for more complex models.

Estimates obtained for model (4.6), using the CSS estimator, are reported in Table F.4, under both normal and stable errors. The estimates indicate a \( d \) value of -0.327, with a standard error of 0.203, when shocks are stable. The normal model yields a higher point estimate of -0.258.\(^{24}\)

The consistency and asymptotic normality of the MLE for the fractional Gaussian ARFIMA model have been established only for \( 0 < d < 0.5 \). However, the Monte Carlo experiments in Cheung and Diebold (1994) (as reported by Sowell (1992b)) suggest that these asymptotics work well even when \( d \) is negative, as is the case here. The stable model yields a LR statistic of 1.34 for testing \( d = 0 \), with a \( p \)-value of 0.247 from a \( \chi^2 \) distribution. The \( p \)-value for this statistic in the normal case exceeds 0.3. The small-sample null distribution for a homoskedastic ARFIMA(3,d,2) model simulated by Sowell (1992a) indicates slightly more density in the tail of this distribution, suggesting that the true \( p \)-values may in fact be higher. Tests based on the Wald statistic also yield similar inferences (Sowell’s simulations show that this statistic also has thicker tails in its small-sample density as compared to a standard normal density). Thus, we are unable to find convincing evidence against the unit root hypothesis versus the fractional alternative,

\[ d \text{ is fractional. When } \eta = 1, \text{ this reduces to the ARFIMA(0,2d,0) process. Chung derives the asymptotic distribution of the CSS estimators of the process parameters, without assuming normality for } \varepsilon_t. \]

\(^{24}\) Sowell (1992a) reports a \( d \) estimate of -0.30 for quarterly real GNP from 1947.I-1989.IV, with a \( t \)-statistic of -1.27, for the ARFIMA(2,d,0) model. Estimation of the normal model, with iid errors, for this period by the CSS procedure reproduced these estimates exactly. Thus, our approximate ML seems to perform fairly well, as compared to exact ML, in terms of estimating the differencing parameter and its standard error.
based on these estimates.\footnote{A test for trend-stationarity versus the fractional alternative could be formulated as a test for $d = -1$. Sowell (1992b) performs this test, and finds similar conclusions for this test and the test for $d = 0$ discussed in the text subsequently.} It should be noted that Sowell’s results, based on his estimated ARFIMA(2,d,0) model without any non-linearities in the mean, also lead to similar inferences regarding $d = 0$ (his LR statistic in this instance is 1.996, with a $p$-value of 0.158).

However, contrary to the above findings, Sowell draws different conclusions regarding the unit root versus fractional alternatives. He bases his conclusions on his estimated ARFIMA(3,d,2) model. Specifically, based on this model, he obtains an LR statistic of 3.046, with a $p$-value of 8.3 percent, and infers that the data is unable to reject a unit root in real GNP at the 95 percent level, although it does reject the null hypothesis at the 90 percent level. Sowell’s justification for using his ARFIMA(3,d,2) parameterisation is based on the minimum AIC criterion, although his results indicate that the SIC selects ARFIMA(2,d,0) in preference to ARFIMA(3,d,2). In any case, it should be noted that the merit of these information criteria for model selection under fractional integration remains to be investigated. Sowell’s estimated model is:

$$
(1 - 1.18L + 0.93L^2 - 0.51L^3)(1 - L)^{-0.59} \Delta y_t = (1 - 0.29L + 0.81L^2) \epsilon_t.
$$

The AR polynomial has one real root at 1.235, and two complex conjugate roots, $0.294 \mu_1.225i$, at a radius of 1.260. The MA polynomial has two complex conjugate roots, $0.179 \mu_1.097i$, at a radius of 1.112. Roots this close to the unit circle and this close to canceling each other out are to be viewed with caution. In any event, it appears that inferences on $d$ are very sensitive to the particular parameterisation chosen. Furthermore, Sowell’s model does not account for any asymmetries in persistence.\footnote{Estimation of (4.6), with $\omega$ restricted to zero, results in a point estimate of $d$ that is closer to zero, a $t$-statistic smaller than one, and a reduced LR test statistic.}

4.6 Concluding Remarks

The main results of our investigations can be summarized as follows. The time series of quarterly real GNP in the post-World War II United States displays significant
leptokurtosis and conditional heteroskedasticity. The observed leptokurtosis cannot be fully accounted for by a GARCH(1,1) model with normal innovations. Instead, as in the case of stock returns, GARCH models with thick-tailed densities are needed. There is statistically significant evidence of asymmetries in the persistence of positive and negative shocks, with negative shocks being less persistent than positive shocks. This asymmetry persists even after accounting for conditional heteroskedasticity and non-normality. With the specific parameterisation of the short run dynamics adopted here, we fail to find significant evidence of long memory.

The finding of significant leptokurtosis in real GNP data accords with the results in Balke and Fomby (1994), who report the presence of outliers in this series. Our results on conditional heteroskedasticity corroborate earlier findings by French and Sichel (1993) and Balke and Fomby (1994). However, the robustness of our results to the presence of outliers contradicts the assertions of Balke and Fomby, who find weakened evidence against homoskedasticity once unusual events are accounted for. Evidence on asymmetries is in tune with the results of Beaudry and Koop (1993) and Potter (1995). The fact that non-linearities become stronger after accounting for big shocks contradicts Scheinkman and LeBaron (1989) and Balke and Fomby (1994), who report weakened evidence against linearity once outliers are taken into account. The presence of asymmetries, together with conditional heteroskedasticity, implies that the finding of non-linearity in the conditional mean is not being driven by changing volatilities, a possibility that Scheinkman and LeBaron demonstrate. Failure to find long memory contradicts the findings of Sowell (1992a), but confirms earlier results by Diebold and Rudebusch (1989).

Our results indicate that linear Gaussian models, and non-linear conditionally heteroskedastic models of the GARCH class, are grossly inadequate for capturing the salient time series features of real GNP. Non-linearities in the conditional mean, and non-normalities beyond what can be accounted for by conditional heteroskedasticity, are important. Measures of persistence of shocks to real income that have been reported in the literature are likely to be biased due to the imposition of symmetry. Our results
suggest that business cycle theorists need to look at more than the covariance properties in order to assess the validity of their models.

There are several ways in which our investigations can be refined. While we have chosen to work with symmetric impulses, it is possible that shocks arrive in an asymmetric fashion. It is conceivable that the propagation mechanism becomes less non-linear, once we account for these asymmetries. Our GARCH model also imposes symmetry, whereas French and Sichel (1993) and Brunner (1992) report increases in conditional variances during recessions. Thus, extensions to exponential GARCH (EGARCH) are likely to be fruitful, especially for short-term forecasting. Further investigations into long memory are called for, especially in light of the sensitivity of results to the parameterisation of short-run dynamics. Since our results indicate the importance of non-linearities, we need to further investigate non-linearities and long memory simultaneously. The econometrics of such models needs to be addressed. Extensions to multivariate frameworks should also prove useful, in order to understand the joint dynamics of various time series.
CONCLUSIONS

We have demonstrated in this study the feasibility of modeling univariately three important economic time series using symmetric stable shocks, namely, the CPI-based U.S. inflation rate, the value-weighted real CRSP stock returns, and the U.S. real GNP. The key to maximum likelihood estimation is the numerical approximation to the symmetric stable density due to McCulloch (1994b).

Inflation rate and stock returns are modeled within a state-space framework. Estimation of the non-Gaussian state-space models is done via numerical integration of the optimal filtering formulae due to Sorenson and Alspach (1971). The method can be readily extended to model other time series of interest. Real GNP is modeled by extending the ARIMA framework to incorporate non-linearities in the specification of the conditional mean function and by allowing for fractional differencing to capture long memory. Estimation of this model is carried out by conditional maximum likelihood.

Some broad conclusions emerge from investigating the three series. Our analysis of the inflation rate suggests statistically significant non-normality of its empirical distribution. Our estimates indicate a characteristic exponent of 1.81, well away from the value of 2 characterizing normal behavior. Our analysis also indicates state-contingent
heteroskedasticity in the series, providing direct evidence on the Friedman hypothesis of a positive relationship between the level of inflation and future uncertainty associated with it. Our analysis demonstrates the relative sluggishness of models with normal shocks in adapting to regime changes as compared to models with fat-tailed stable disturbances. Moreover, models with stable shocks rightly attribute greater uncertainty during confusing episodes as when outliers or regime shifts occur. Accounting for non-normality results in more efficient estimation of model parameters and tighter prediction intervals. The univariate models developed for inflation can be used for forecasting and to simulate historical inflation forecasts conditional on the history of inflation.

Regarding stock returns our principal findings are as follows. The monthly CRSP value-weighted real stock returns for the post-1952 period display significant leptokurtosis. The estimated value of the characteristic exponent is about 1.89. Stock returns exhibit significant persistence in volatility that can be well-described by a GARCH-like process. However, a GARCH process, with normal innovations, cannot fully account for the observed leptokurtosis in the series, as claimed in some prior studies. Even the conditional densities are leptokurtic enough to warrant using stable shocks. Although we do not find evidence of any seasonals in mean returns, the months of January and March exhibit statistically significant seasonals in volatility. Our results concerning the predictability of monthly stock returns are negative. Our state-space models fail to reveal a statistically significant persistent signal in returns, after taking into account the non-normality, seasonality, and volatility persistence in the series.

With regard to real GNP the main results of our investigations can be summarized as follows. The time series of quarterly real GNP in the post-World War II United States displays significant leptokurtosis and conditional heteroskedasticity. The observed leptokurtosis cannot be fully accounted for by a GARCH(1,1) model with normal innovations. Instead, as in the case of stock returns, GARCH models with thick-tailed densities are needed. There is statistically significant evidence of asymmetries in the persistence of positive and negative shocks to real income. Specifically, negative shocks are less persistent than positive shocks. This asymmetry persists even after accounting for
conditional heteroskedasticity and non-normality of the series. Failure to account for non-linearities could result in inaccurate estimates of the persistence factors, although non-normalities and/or conditional heteroskedasticity do not seem very important in this respect. With the specific parameterization of the short run dynamics adopted in the paper, we fail to find significant evidence of long memory in the data.

In summary, the study shows that non-normality is prevalent in all the three time series examined, suggesting that this may be a more pervasive feature of economic data than has hitherto been acknowledged. The assumption of iid disturbances also seems misplaced. Dependence in second moments seems to be the rule. Conditional heteroskedasticity, either GARCH-like or state-contingent, is present in all three series. These results point to the general inadequacy of homoskedastic linear models and non-linear conditionally heteroskedastic models, driven by normal shocks, for describing these time series.

A few shortcomings of our methods can be noted. We have generally ignored the asymmetric response of volatility to positive and negative shocks, whereas researchers have noted such asymmetries in the empirical distribution of these series. Incorporating such asymmetric effects seems useful. We have restricted ourselves to symmetrically distributed shocks. However, there is some evidence in the literature that shocks arrive in an asymmetric fashion. It would be useful to extend our work to deal with asymmetric stable shocks. However, a fast numerical approximation to the skew-symmetric stable shocks needs to be developed first. Our univariate methods limit exploration of interesting economic interrelationships between variables. It would be interesting to generalize to multivariate models. However, the numerical integration technique used in Chapters 2 and 3 becomes unstable under large dimensionality of the state vector. Also the computational expenses involved may be prohibitive. Monte Carlo integration techniques may prove more useful in this regard.
APPENDIX A

STABLE DISTRIBUTIONS

A stable distribution may be defined in terms of its log-characteristic function as:

\[ \ln \mathbb{E} \exp(iXt) = i\delta t + \psi_{\alpha,\beta}(ct) \]  

(A.1)

where \( \psi_{\alpha,\beta}(t) = \begin{cases} 
|t|^\alpha \left(1 - i\beta \tan(\pi\alpha / 2) t/|t|\right) & \text{for } \alpha \neq 1 \\
-|t|(1 + i\beta \ln(|t|)2t / (\pi|t|)) & \text{for } \alpha = 1. 
\end{cases} \)  

(A.2)

The parameters \( c > 0 \) and \( \delta \in (-\infty, \infty) \) are measures of scale and location respectively, \( \beta \in [-1,1] \) is the skewness parameter with \( \beta = 0 \) indicating symmetry and \( \beta > 0 \) indicating positive skewness, and \( \alpha \in (0,2] \) is the characteristic exponent governing the tail behavior, with a smaller value of \( \alpha \) indicating thicker tails. Thus, the standardized stable random variable, with \( \delta = 0 \) and \( c = 1 \), has the log-characteristic function \( \psi_{\alpha,\beta}(t) \).

Approximately 50 percent of the density lies within \( \pm c \) of \( \delta \).

Some familiar cases of the stable distributions are the normal distribution with \( \alpha = 2 \) and the Cauchy distribution with \( \alpha = 1 \) and \( \beta = 0 \). Moments of order \( k \geq \alpha \) do not exist unless \( \alpha = 2 \); thus the normal distribution is the only member with finite variance, equal to \( 2c^2 \). The stable distribution and density may be evaluated by using
Zolotarev’s (1986, p.74,78) proper integral representations or by taking the inverse Fourier transform of the characteristic function. McCulloch (1994b) has developed a fast numerical approximation to the symmetric stable distribution and density that has an expected relative density precision of $10^{-6}$ for $\alpha \in [0.84,2]$. We therefore restrict ourselves in this paper to stable distributions with $\beta = 0$ and $\alpha$ in this range for computational convenience.

Only stable random variables have the convenient and useful statistical properties that they have domains of attraction (generalized central limit theorem) and that they belong to their own domain of attraction (stability). If $X_1$ and $X_2$ are independent stable random variables with parameters $(\alpha, \beta_1, \delta_1, c_1)$ and $(\alpha, \beta_2, \delta_2, c_2)$ respectively then $Y = X_1 + X_2$ is stable with parameters $(\alpha, \beta, \delta, c)$ such that

\begin{align*}
  c^\alpha &= c_1^\alpha + c_2^\alpha \quad \text{(A.3)} \\
  \beta &= (\beta_1 c_1^\alpha + \beta_2 c_2^\alpha) / c^\alpha \quad \text{(A.4)} \\
  \delta &= \begin{cases} 
    \delta_1 + \delta_2 & \text{for } \alpha \neq 1 \\
    \delta_1 + \delta_2 + 2(\beta c \ln(c) - \beta_1 c_1 \ln(c_1) - \beta_2 c_2 \ln(c_2)) / \pi & \text{for } \alpha = 1.
  \end{cases} \quad \text{(A.5)}
\end{align*}
APPENDIX B

SORENSON-ALSPACH FILTERING EQUATIONS AND THEIR NUMERICAL IMPLEMENTATION

B.1: Sorenson-Alspach filtering equations

Let \( y_t, t = 1, \ldots, T \), be an observed time series and \( x_t \) an unobserved state variable, stochastically determining \( y_t \). Denote \( Y_t = (y_1, \ldots, y_t) \). The recursive formulae for obtaining one-step ahead prediction and filtering densities, due to Sorenson and Alspach (1971), are as follows:

\[
\begin{align*}
    p(x_t | Y_{t-1}) &= \int_{-\infty}^{\infty} p(x_t | x_{t-1}) p(x_{t-1} | Y_{t-1}) dx_{t-1}, \\
    p(x_t | Y_t) &= p(y_t | x_t) p(x_t | Y_{t-1}) / p(y_t | Y_{t-1}), \\
    p(y_t | Y_{t-1}) &= \int_{-\infty}^{\infty} p(y_t | x_t) p(x_t | Y_{t-1}) dx_t.
\end{align*}
\]

Finally, the log-likelihood function is given by:

\[
\log p(y_1, \ldots, y_T) = \sum_{t=1}^{T} \log p(y_t | Y_{t-1}).
\]
These formulae have been applied to non-Gaussian data and extended to include a smoother formula by Kitagawa (1987). When shocks are normal ($\alpha = 2$ in our models), this filter collapses to the Kalman filter.

**B.2: Numerical implementation of filtering equations I**

The Sorenson-Alspach (1971) filter and predictive densities were evaluated at a grid of 100 points equally spaced on a truncated portion of the real line (for two of the models, Model 2a-S and Model 2-N in Chapter 2, we used 200 nodes because 100 nodes were not sufficient to produce convergence of maximum likelihood estimation). The left truncation point was chosen to lie 4 standard deviations (of the measurement shock $\varepsilon$, as measured by a preliminary Kalman filter) below the minimum observed inflation rate (i.e. at -18.0%) and the right truncation point 4 standard deviations above the maximum observed inflation (i.e. at 32.0%). The likelihood and the predictive density integrals (equations (2.2c) and (2.2a), or equivalently (B.3) and (B.1), resp.) were evaluated numerically by a piecewise cubic quadrature technique, as follows: Integration between any two interior nodes was performed by fitting a piecewise cubic function through the four nearest nodes and approximating the required area under the integrand between those nodes by the area under the cubic. The outermost intervals employ the same cubics as the adjacent intervals. For equispaced nodes, 8 or more in number, this quadrature procedure yields the weights $8/24$, $31/24$, $20/24$, $25/24$, $1$, ..., $1$, $25/24$, $20/24$, $31/24$, $8/24$ for the ordinates. The numerically computed predictive density was normalized in order to ensure that it integrated to unity. The piecewise linear interpolation and the trapezoidal rule for integration suggested by Kitagawa (1987) was not employed. Hodges and Hale (1993) propose an integration by parts procedure to speed up the Kitagawa procedure, but this was not employed either.

The filter is initialized by the diffuse prior method since we are assuming that the process is nonstationary, i.e., $p(x_1|Y_1) = S_\alpha(y_i; x_1, c_{\varepsilon_i}(x_1))$, where $S_\alpha(x;\delta,c)$ is the symmetric stable density. Starting points for the hyperparameter estimation are obtained from the Kalman filter under normality for the homoskedastic model and by pooled
maximum likelihood estimation (see McCulloch (1994a)) for the conditionally heteroskedastic models.

The accuracy of our numerical quadrature can be gauged by a comparison of the log-likelihood value for Model 1 in Chapter 2 obtained from our numerical integration with $\alpha$ restricted to be 2, with that obtained from the Kalman filter, for given values of the other hyperparameters. We verified that with 100 nodes our numerical approximation gives log-likelihood values accurate to within 0.001 at the estimated hyperparameters of Model 1-N. In light of this, our numerical integration appears to be sufficiently accurate for drawing valid inferences from data. Figures D.1 and D.4 are constructed by linearly connecting the estimated densities at the computed nodes. Calculations were performed in GAUSS 3.2.1 on a (corrected) Pentium personal computer.

**B.3: Numerical implementation of filtering equations II**

The Sorenson-Alspach (1971) filter and predictive densities were evaluated at a grid of 100 points equally spaced on a truncated portion of the real line. The left truncation point was chosen to lie 4 standard deviations (of the measurement shock $\varepsilon$, as measured by a preliminary Kalman filter) below the minimum observed real return (i.e. at -15.6%) and the right truncation point 4 standard deviations above the maximum observed return (i.e. at 25.9%). The likelihood and the predictive density integrals (equations (B.3) and (B.1) resp.) were evaluated numerically by a piecewise cubic quadrature technique, as follows: Integration between any two interior nodes was performed by fitting a piecewise cubic function through the four nearest nodes and approximating the required area under the integrand between those nodes by the area under the cubic. The outermost intervals employ the same cubics as the adjacent intervals. For equispaced nodes, 8 or more in number, this quadrature procedure yields the weights $8/24, 31/24, 20/24, 25/24, 1, 1, ..., 1, 25/24, 20/24, 31/24, 8/24$ for the ordinates. The numerically computed predictive density was normalized in order to ensure that it integrated to unity. The piecewise linear interpolation and the trapezoidal rule for integration suggested by Kitagawa (1987) was not employed. Hodges and Hale
(1993) propose an integration by parts procedure to speed up the Kitagawa procedure, but this was not employed either.

In model (3.7), \( r_t \) is the observed series, \( p(r_t | x_t) = S_\alpha (r_t - x_t; 0, C_t c_t) \) and \( p(x_t | x_{t-1}) = S_\alpha (x_t - \mu - \phi (x_{t-1} - \mu); 0, C_s c_{t-1} c_t) \), where \( S_\alpha (x; \delta, c) \) is the symmetric stable density. The filter is initialized by the unconditional distribution of the state variable since the process for returns is stationary, i.e.

\[
p(x_0 | Y_0) = S_\alpha (x_0; \mu, c_{x_0}), c_{x_0} = C_s c_{\eta} c_0 / (1 - \phi^\alpha)^{1/\alpha}
\]

where \( c_0 \) is the unconditional mean of \( c_t \) which evolves according to the process (3.7c). Starting points for the hyperparameter estimation are obtained from the Kalman filter under normality.

The accuracy of our numerical quadrature can be gauged by a comparison of the log-likelihood value for model (3.7) obtained from our numerical integration with \( \alpha \) restricted to be 2, with that obtained from the Kalman filter, for given values of the other hyperparameters. We verified that with 100 nodes our numerical approximation gives log-likelihood values accurate to four decimal places at the estimated hyperparameters of the normal model (3.7). In light of this our numerical integration appears to be sufficiently accurate for drawing valid inferences from data. Calculations were performed in GAUSS 3.2.1 on a (corrected) Pentium personal computer.
INFLATION DATA AND SEASONAL ADJUSTMENT

Our measure of inflation is based primarily on the CPI-U. However, the CPI-U is generally recognized to have seriously mismeasured the housing component of the true cost of living prior to its 1983 revision. The Bureau of Labor Statistics (BLS) CPI-X series recomputes the CPI-U on the post-1983 basis and is available from June 1967 until its adoption as the official CPI-U in 1983. We therefore constructed a CPI-UX series by linking together the CPI-U from 1953:10 to 1967:6, the CPI-X from 1967:6 to 1983:1, and again the CPI-U from 1983:1 to 1993:9. The 1967 base year was used for both series, so as to reduce rounding error. During 1967:6 to 1983:1, the CPI-U and CPI-X diverge by 8.5 percentage points.

The inflation rate was constructed as the first differences of the natural logarithms of the constructed CPI-UX series, expressed as annualized percentage rates. This gave 479 inflation observations, from 1953:11 (Oct-Nov inflation) to 1993:9. The 1953 starting date was so chosen because of the extensive revisions that were made to the process of constructing the CPI in the early 1950s (see Cosimano and Jansen (1988) for a brief discussion and some evidence for structural change in the inflation process).
Conventional, Gaussian-based augmented Dickey-Fuller tests failed to reject a unit root in the inflation rate at the 10 percent significance level against the alternatives of a stationary and a trend stationary process when the AIC was used to select the number of lags in the regression. When the test was repeated with lags lengths ranging from one to eleven, we were unable to reject a unit root when the lag length exceeded 8 in the case of the stationary alternative and 5 in the case of the trend stationary alternative. The low power of the unit root tests being well-known, we assume here that inflation is difference stationary. Previous work on modeling univariate inflation process (e.g. Barsky (1987), Brunner and Hess (1993), Hafer and Hein (1990)) has typically used the ARIMA(0,1,1) representation, implicitly treating inflation as difference stationary.

A plot of the sample autocorrelations of the first differences of the level of inflation (not shown) indicates quantitatively significant autocorrelations at seasonal lags, suggesting that seasonality may be important in the series. To deal with this, we set up an unobserved components model for observed inflation $y_t$, expressed as the sum of a trend component $x_t$, a seasonal $S_t$, and an irregular noise $\varepsilon_t$, as follows:

$$y_t = x_t + \sum_{i=1}^{12} S_i I_{it} + \varepsilon_t, \quad \varepsilon_t \sim \text{iid } N(0, \sigma^2_{\varepsilon}) \quad (C.1)$$

$$x_t = x_{t-1} + \eta_t, \quad \eta_t \sim \text{iid } N(0, \sigma^2_{\eta}) \quad (C.2)$$

Here, the indicator function $I_{it} = 1$ if $t$ corresponds to month $i$ and zero otherwise, and the monthly seasonals are constrained to sum to zero. We estimate the model using the Kalman filter. We also estimate a restricted version of the model, where the seasonal dummies are all set to zero.

Table C.1 gives the values of the individual seasonals along with their $t$-values, and the results of the likelihood ratio test for their joint significance. The likelihood ratio test easily rejects the hypothesis that the seasonals are all zero at the 5 percent significance level. The table also indicates that the biggest seasonal occurs in the month of October (1.22%) and the smallest in April (-1.57%). The $t$-values show that only the seasonals in March (-1.18%), April and October are individually statistically significant at 5 percent level. While one could reestimate the above model by including seasonal...
dummies in only these three months, we do not do so here. Instead, we simply
deseasonalise the data by subtracting the seasonals obtained above from the raw series,
and conduct the remaining analysis using the resulting seasonally adjusted inflation.

We seasonally adjusted the data using constant seasonal dummies under the
assumption of normality. Ideally, a structural time series model composed of trend,
seasonal and irregular components should be set up and simultaneously estimated with
stable disturbances. Although it is fairly straightforward to simultaneously estimate
multiple components in state-space models under normality, the generalized filtering
technique of Sorenson and Alspach (1971) is not easily implementable numerically if the
dimension of the state vector is large. Although we only considered constant seasonals,
time-varying seasonals could also be entertained.

The raw CPI-U and CPI-X data, the constructed CPI-UX, raw inflation and
seasonally adjusted series, as well as the predicted inflation data of Table D.4, may be
obtained by anonymous FTP to ecolan.sbs.ohio-state.edu/pub/PVBJHM1, or from the
authors on request.
<table>
<thead>
<tr>
<th>Month</th>
<th>Seasonal Dummies</th>
<th>t-stat</th>
<th>Model with no seasonals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jan</td>
<td>0.516 (0.345)</td>
<td>1.497</td>
<td></td>
</tr>
<tr>
<td>Feb</td>
<td>-0.251 (0.348)</td>
<td>-0.720</td>
<td></td>
</tr>
<tr>
<td>Mar</td>
<td>-1.176 (0.344)</td>
<td>-3.415</td>
<td></td>
</tr>
<tr>
<td>Apr</td>
<td>-1.570 (0.344)</td>
<td>-4.562</td>
<td></td>
</tr>
<tr>
<td>May</td>
<td>-0.119 (0.344)</td>
<td>-0.347</td>
<td></td>
</tr>
<tr>
<td>Jun</td>
<td>0.572 (0.344)</td>
<td>1.664</td>
<td></td>
</tr>
<tr>
<td>Jul</td>
<td>0.262 (0.344)</td>
<td>0.763</td>
<td></td>
</tr>
<tr>
<td>Aug</td>
<td>0.413 (0.344)</td>
<td>1.201</td>
<td></td>
</tr>
<tr>
<td>Sep</td>
<td>0.294 (0.344)</td>
<td>0.855</td>
<td></td>
</tr>
<tr>
<td>Oct</td>
<td>1.223 (0.344)</td>
<td>3.555</td>
<td></td>
</tr>
<tr>
<td>Nov</td>
<td>0.329 (0.344)</td>
<td>0.956</td>
<td></td>
</tr>
<tr>
<td>Dec</td>
<td>-0.494 (0.345)</td>
<td>-1.433</td>
<td></td>
</tr>
</tbody>
</table>

LR statistic for no seasonals: 46.85 (χ²11 at 5% = 19.68)

logL = -1113.159  -1136.583

Standard errors are in parentheses.
Seasonals are constrained to sum to zero.
Jan seasonal is for Dec-Jan inflation, etc.

Table C.1: Inflation seasonals
APPENDIX D

TABLES AND FIGURES RELATED TO CHAPTER 2
### Table D.1: Estimation results - Stable model

<table>
<thead>
<tr>
<th>Model</th>
<th>$\alpha$</th>
<th>$c_c$</th>
<th>$c_\eta$</th>
<th>$\theta_c$</th>
<th>$\theta_\eta$</th>
<th>logL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1-S</td>
<td>1.803</td>
<td>1.369</td>
<td>0.278</td>
<td>-1100.78</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.094)</td>
<td>(0.077)</td>
<td>(0.064)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model 2-S</td>
<td>1.806</td>
<td>1.262</td>
<td>0.114</td>
<td>1.167</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.102)</td>
<td>(0.111)</td>
<td>(0.069)</td>
<td>(-1.504)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model 2a- S*</td>
<td>1.808</td>
<td>1.349</td>
<td>0.109</td>
<td>0.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.100)</td>
<td>(0.078)</td>
<td>(0.066)</td>
<td>(-1.781)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model 2b-S</td>
<td>1.806</td>
<td>1.207</td>
<td>0.279</td>
<td>0.000</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.119)</td>
<td>(0.134)</td>
<td>(0.068)</td>
<td>(0.053)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Standard errors are in parentheses.

* Preferred unrestricted model.

### Table D.2: Estimation results - Normal model ($\alpha = 2$)

<table>
<thead>
<tr>
<th>Model</th>
<th>$c_c$</th>
<th>$c_\eta$</th>
<th>$\theta_c$</th>
<th>$\theta_\eta$</th>
<th>logL</th>
<th>$2\Delta \log L$ for $\alpha = 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model 1-N</td>
<td>1.563</td>
<td>0.357</td>
<td>-1113.16</td>
<td>24.76</td>
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<tr>
<td></td>
<td>(0.029)</td>
<td>(0.051)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model 2-N</td>
<td>1.332</td>
<td>0.188</td>
<td>0.090</td>
<td>0.643</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.101)</td>
<td>(0.073)</td>
<td>(0.050)</td>
<td>(0.690)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model 2a-N</td>
<td>1.550</td>
<td>0.160</td>
<td>1.018</td>
<td>-1108.91</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.058)</td>
<td>(0.081)</td>
<td>(1.292)</td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Model 2b-N</td>
<td>1.303</td>
<td>0.361</td>
<td>0.106</td>
<td>-1108.26</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(0.107)</td>
<td>(0.056)</td>
<td>(0.056)</td>
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<td></td>
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<tr>
<td>Model 3-N**</td>
<td>1.283</td>
<td>0.290</td>
<td>0.119</td>
<td>-1106.84</td>
<td></td>
<td></td>
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<tr>
<td></td>
<td>(0.099)</td>
<td>(0.051)</td>
<td>(0.055)</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

Standard errors are in parentheses.

* 0.005 critical value = 7.664 for homoskedastic model with estimated constant location parameter, $T = 300$ (McCulloch 1996a, Table 4b).

** Preferred Gaussian model.
Table D.3: Tests for heteroskedasticity

<table>
<thead>
<tr>
<th>Null vs. Alternative</th>
<th>$2 \Delta \log L$ Stable model</th>
<th>$2 \Delta \log L$ Gaussian model</th>
<th>5% critical value</th>
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</thead>
<tbody>
<tr>
<td>Model 1 vs. Model 2</td>
<td>9.64</td>
<td>14.52</td>
<td>5.99</td>
</tr>
<tr>
<td>Model 2a vs. Model 2</td>
<td>1.02</td>
<td>6.02</td>
<td>3.84</td>
</tr>
<tr>
<td>Model 2b vs. Model 2</td>
<td>6.98</td>
<td>4.72</td>
<td>3.84</td>
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<tr>
<td>Model 3 vs. Model 2</td>
<td>4.20</td>
<td>1.88</td>
<td>3.84</td>
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<tr>
<td>Model 2a vs. Model 3</td>
<td>3.18</td>
<td>4.14</td>
<td>3.84</td>
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<tr>
<td>month</td>
<td>mean</td>
<td>s.e.</td>
<td>month</td>
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<tr>
<td>-------</td>
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<td>------</td>
<td>-------</td>
</tr>
<tr>
<td>53.11</td>
<td>-3.26</td>
<td>∞</td>
<td>57.03</td>
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<tr>
<td>53.12</td>
<td>-1.57</td>
<td>1.40</td>
<td>57.04</td>
</tr>
<tr>
<td>54.01</td>
<td>0.01</td>
<td>1.25</td>
<td>57.05</td>
</tr>
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<td>54.02</td>
<td>-0.61</td>
<td>1.04</td>
<td>57.06</td>
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<td>54.03</td>
<td>-0.87</td>
<td>0.94</td>
<td>57.07</td>
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<td>54.04</td>
<td>-1.40</td>
<td>0.92</td>
<td>57.08</td>
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<td>54.05</td>
<td>-0.52</td>
<td>0.90</td>
<td>57.09</td>
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<td>54.06</td>
<td>-0.38</td>
<td>0.79</td>
<td>57.10</td>
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<td>54.07</td>
<td>-0.36</td>
<td>0.73</td>
<td>57.11</td>
</tr>
<tr>
<td>54.08</td>
<td>-0.44</td>
<td>0.70</td>
<td>57.12</td>
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<td>54.09</td>
<td>-0.85</td>
<td>0.76</td>
<td>58.01</td>
</tr>
<tr>
<td>54.10</td>
<td>-1.13</td>
<td>0.79</td>
<td>58.02</td>
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Table D.4: Stable filter inflation forecasts

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Fig.D.1 Filter densities from the stable model (Jan only)
Fig.D.2 Filter estimates - A. Stable, B. Normal
Fig.D.3 Standard errors from the filter densities
Fig. D.4 Filter densities, $p(x_i|Y_i)$ —
(__ Stable, -.- Gaussian, --- newly observed inflation)
(Fig. D.4 contd.)

B

(contd.)
(Fig. D.4 contd.)
APPENDIX E

TABLES AND FIGURES RELATED TO CHAPTER 3
\[ r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim \text{iid } N(0, 2c^2) \]  \hspace{1cm} (3.1)

\[ r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim \text{iid } S_{\alpha}(0, c) \]  \hspace{1cm} (3.2)

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<td>2 (restricted) 1.833 ( (0.060) )</td>
<td>1.833 ( (0.060) )</td>
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<td>( c )</td>
<td>2.982 (0.094) 2.652 ( (0.111) )</td>
<td>2.982 (0.094) 2.652 ( (0.111) )</td>
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<tr>
<td>( \mu )</td>
<td>0.558 (0.188) 0.677 ( (0.177) )</td>
<td>0.558 (0.188) 0.677 ( (0.177) )</td>
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<td>( 2\Delta \log L ) for ( \alpha = 2 )</td>
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Hessian-based standard errors are in parentheses.

* 0.005 critical value equals 6.688 for \( T=1000 \), and 7.664 for \( T=300 \) (McCulloch (1996), Table 4, panel b).

**Table E.1: Test for iid normality**
Model \( V1: r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim S_\alpha (0, c_t) \)
\[ (3.4) \]
Model \( V2: r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim S_\alpha (0, c_t), \quad c_t = c, \forall \text{ months } \neq \text{ Jan and Mar} \)
Model \( V3: r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim S_\alpha (0, c_t), \quad c_t = c, \forall \text{ months } \neq \text{ Jan} \)
Model \( V3: r_t = \mu + \varepsilon_t, \quad \varepsilon_t \sim S_\alpha (0, c_t), \quad c_t = c, \forall \text{ months } \neq \text{ Mar} \)

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<th>Model V3</th>
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<td>1.894 (0.042)</td>
<td>1.856 (0.063)</td>
<td>1.840 (0.056)</td>
<td>1.852 (0.051)</td>
<td>2 (restricted)</td>
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<tr>
<td>( \mu )</td>
<td>0.680 (0.174)</td>
<td>0.699 (0.171)</td>
<td>0.674 (0.174)</td>
<td>0.706 (0.173)</td>
<td>0.542 (0.185)</td>
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<td>( c )</td>
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<td>2.560 (0.109)</td>
<td>2.736 (0.114)</td>
<td>2.953 (0.102)</td>
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<tr>
<td>( c_{\text{Jan}} )</td>
<td>3.670 (0.434)</td>
<td>3.653 (0.443)</td>
<td>3.647 (0.446)</td>
<td>3.723 (0.412)</td>
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<td>( c_{\text{Feb}} )</td>
<td>2.258 (0.259)</td>
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<td>( c_{\text{Mar}} )</td>
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<td>1.888 (0.231)</td>
<td>2.396 (0.262)</td>
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Hessian-based standard errors are in parentheses.

**Table E.2: Seasonal volatility models**

97
Null vs Alternative Model (Null hypothesis) | $2\Delta \log L$ | 2.5% critical values
--- | --- | ---
Model (3.2) vs Model (3.4) (Identical scales across all months) | 23.26 | 21.92
Model V1 vs Model (3.4) (all months have identical scales, except Jan and Mar) | 9.35 | 19.02
Model (3.2) vs Model V1 (Jan and Mar are no different) | 13.92 | 7.38
Model V2 vs Model V1 (Mar is no different) | 5.72 | 5.02
Model V3 vs Model V1 (Jan is no different) | 6.98 | 5.02

Table E.3: Tests for seasonals in scales

| Seasonal GARCH | $r_t = \mu + \varepsilon_t$, $\varepsilon_t \sim S_\alpha(0, C_s, c_t)$ | (3.6a) |
| Non-seasonal GARCH | $r_t = \mu + \varepsilon_t$, $\varepsilon_t \sim S_\alpha(0, c_t)$ | GARCH model, G: $c_t^\alpha = \omega + \beta c_{t-1}^\alpha + \delta |r_{t-1} - \mu|^\alpha$ | (3.6b) |

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<th>Normal Model (3.6)</th>
<th>Normal Model G</th>
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<td>1.874 (0.065)</td>
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<td>2 (restricted)</td>
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<td>$\mu$</td>
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<td>0.743 (0.171)</td>
<td>0.577 (0.175)</td>
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<td>$w$</td>
<td>0.720 (0.340)</td>
<td>0.774 (0.400)</td>
<td>0.552 (0.275)</td>
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<td>$\beta$</td>
<td>0.783 (0.064)</td>
<td>0.793 (0.076)</td>
<td>0.851 (0.041)</td>
<td>0.857 (0.044)</td>
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<td>$\delta$</td>
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Hessian-based standard errors are in parentheses.

Table E.4: Persistence in volatility (constant mean)
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Table E.5: Tests for GARCH effects (under stable errors)

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0.005 critical value equals 6.688 for $T=1000$, and 7.664 for $T=300$ (McCulloch (1996), Table 4, panel b).

Table E.6: Tests for normality ($\alpha = 2$)
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$$r_t = x_t + \varepsilon_t, \quad \varepsilon_t \sim S_\alpha (0, C_s \varepsilon_t) \quad (3.7a)$$

$$(x_t - \mu) = \phi (x_{t-1} - \mu) + \eta_t, \quad \eta_t \sim S_\alpha (0, C_s \eta_t) \quad (3.7b)$$

$$c_t^\alpha = \omega + \beta c_{t-1}^\alpha + \delta (r_{t-1} - E(r_{t-1}\mid r_1, r_2, \ldots, r_{t-2})) / C_s^\alpha. \quad (3.7c)$$

<table>
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<tr>
<th>Parameters</th>
<th>Model (3.7)</th>
<th>Model (3.6)</th>
<th>Normal model (3.7)</th>
<th>Normal model (3.6)</th>
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<td>1.892 (0.065)</td>
<td>2 (restricted)</td>
<td>2 (restricted)</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.926 (0.119)</td>
<td>0.764 (0.164)</td>
<td>0.592 (0.197)</td>
<td>0.577 (0.175)</td>
</tr>
<tr>
<td>$w$</td>
<td>0.619 (0.291)</td>
<td>0.720 (0.340)</td>
<td>0.478 (0.243)</td>
<td>0.552 (0.275)</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.792 (0.059)</td>
<td>0.783 (0.064)</td>
<td>0.851 (0.039)</td>
<td>0.851 (0.041)</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.052 (0.017)</td>
<td>0.049 (0.017)</td>
<td>0.050 (0.018)</td>
<td>0.043 (0.015)</td>
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<tr>
<td>$c_{Jan}$</td>
<td>1.412 (0.176)</td>
<td>1.425 (0.178)</td>
<td>1.304 (0.156)</td>
<td>1.301 (0.153)</td>
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<td>$c_{Mar}$</td>
<td>0.625 (0.091)</td>
<td>0.657 (0.092)</td>
<td>0.751 (0.099)</td>
<td>0.794 (0.093)</td>
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<tr>
<td>$c_\eta$</td>
<td>0.098 (0.045)</td>
<td>0.240 (0.175)</td>
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<td>$\phi$</td>
<td>0.424 (0.300)</td>
<td>0.631 (0.253)</td>
<td>0.631 (0.253)</td>
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</table>

| logL       | -1403.605 | -1405.317 | -1422.235 | -1423.397 |
| $2\Delta \log L^*$ for $\phi = 0$ | 3.42 | 2.32 | 2.32 | 2.32 |
| $2\Delta \log L^{**}$ for $\alpha = 2$ | 37.26 | 36.16 | 36.16 | 36.16 |

Hessian-based standard errors are in parentheses.

* 10 percent critical value for a $\chi^2_1$ equals 3.841, and for a $\chi^2_2$ equals 5.991.

** 0.005 critical value equals 6.688 for T=1000, and 7.664 for T=300 (McCulloch (1996), Table 4, panel b).

Table E.8: Persistence in mean returns and volatility
Fig.E.1 Monthly value-weighted CRSP real stock returns
Fig.E.2 Stock return volatility from the seasonal GARCH(1,1) model - A. Stable, B. Normal
Fig.E.3 Real CRSP returns and filter mean
APPENDIX F

TABLES AND FIGURES RELATED TO CHAPTER 4
\begin{align*}
(1 - \phi_1 L - \phi_2 L^2)(\Delta y_t - \mu) &= \varepsilon_t, \quad \varepsilon_t \sim \text{iid } N(0,c^2) & (4.1) \\
(1 - \phi_1 L - \phi_2 L^2)(\Delta y_t - \mu) &= \varepsilon_t, \quad \varepsilon_t \sim \text{iid } S_{\alpha}(0,c) & (4.2)
\end{align*}

\begin{table}[h]
\centering
\begin{tabular}{lcc}
\hline
\(\alpha\) & 2 (restricted) & 1.817 (0.129) \\
\hline
\(\mu\) & 0.772 (0.119) & 0.778 (0.108) \\
\(\phi_1\) & 0.345 (0.072) & 0.363 (0.071) \\
\(\phi_2\) & 0.096 (0.072) & 0.055 (0.075) \\
c & 0.650 (0.033) & 0.583 (0.043) \\
log-L & -253.491 & -250.439 \\
LR statistic for \(\alpha = 2\) & & 6.10 \\
& & (1 percent critical value = 4.76*)
\hline
\end{tabular}
\caption{Homoskedastic ARIMA(2,1,0) model}
\end{table}

Standard errors appear in parentheses.
* These are Monte Carlo small-sample critical values derived from McCulloch (1996) (table 4, panel b) who tabulates these for a simple location model. All other tests of \(\alpha = 2\), reported in the other tables, are also based on these critical values.
\[(1 - \phi_1 L - \phi_2 L^2)(\Delta y_t - \mu) = \varepsilon_t, \quad \varepsilon_t \sim S_\alpha(0, c_t)\] \hspace{1cm} (4.3a)

\[c_t^\alpha = b_1 + b_2 c_{t-1}^\alpha + b_3 |\varepsilon_{t-1}|^\alpha\] \hspace{1cm} (4.3b)

<table>
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<tr>
<th>(\alpha)</th>
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<tr>
<td>(\mu)</td>
<td>0.729 (0.106)</td>
<td>0.764 (0.099)</td>
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<tr>
<td>(\phi_1)</td>
<td>0.348 (0.076)</td>
<td>0.336 (0.077)</td>
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<td>(\phi_2)</td>
<td>0.144 (0.076)</td>
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<tr>
<td>(b_1)</td>
<td>6.0e-08 (1.1e-05)</td>
<td>0.014 (0.019)</td>
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<tr>
<td>(b_2)</td>
<td>0.898 (0.030)</td>
<td>0.801 (0.093)</td>
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<tr>
<td>(b_3)</td>
<td>0.049 (0.019)</td>
<td>0.082 (0.037)</td>
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<tr>
<td>(c_0)</td>
<td>1.105 (0.309)</td>
<td>0.313 (0.241)</td>
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<td>log-L</td>
<td>-243.991</td>
<td>-242.151</td>
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<tr>
<td>LR statistic for (\alpha = 2)</td>
<td>3.68</td>
<td>(2 percent critical value = 2.93)</td>
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<tr>
<td>LR statistic for homoskedasticity*</td>
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Standard errors appear in parentheses.

* This is a essentially a test of \(b_2 = b_3 = 0\). Under this null, \(b_1\) and \(c_0\) are trivial transforms of one another. Therefore, it is not clear whether the LR statistic has an asymptotic \(\chi^2\) distribution with two or three degrees of freedom. The test rejects, however, at better than the 0.005 level using either distribution.

Table F.2: Heteroskedastic ARIMA(2,1,0)-GARCH(1,1) model
\[ (1 - \phi_1 L - \phi_2 L^2)(\Delta y_t - \mu) = \omega \cdot \text{CDR}_{t-1} + \varepsilon_t, \varepsilon_t \sim S_\alpha (0, c_t) \] 

(4.5a)

\[ c_t^\alpha = b_1 + b_2 c_{t-1} + b_3 |\varepsilon_{t-1}|^\alpha \]

(4.5b)

<table>
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<tr>
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<tr>
<td>$\mu$</td>
<td>0.629 (0.137)</td>
<td>0.609 (0.131)</td>
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<td>$\phi_1$</td>
<td>0.336 (0.075)</td>
<td>0.330 (0.073)</td>
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<td>$\phi_2$</td>
<td>0.217 (0.084)</td>
<td>0.198 (0.082)</td>
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<td>$b_1$</td>
<td>8.3e-06 (9.9e-04)</td>
<td>0.004 (0.011)</td>
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<td>$b_2$</td>
<td>0.907 (0.029)</td>
<td>0.901 (0.063)</td>
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<tr>
<td>$b_3$</td>
<td>0.042 (0.017)</td>
<td>0.037 (0.022)</td>
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<td>$c_0$</td>
<td>1.086 (0.283)</td>
<td>0.798 (0.334)</td>
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<td>$\omega$</td>
<td>0.221 (0.108)</td>
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<td>log-L</td>
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<tr>
<td>LR statistic for $\alpha = 2$</td>
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<td>LR statistic for $\omega = 0^*$</td>
<td>4.10</td>
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Standard errors appear in parentheses.

* Five percent critical value for a $\chi^2_1$ equals 3.841, and 2.5 percent critical value equals 5.024.

Table F.3: Heteroskedastic non-linear model
\[(1 - \phi_1 L - \phi_2 L^2)(1 - L)^d (\Delta y_t - \mu) = \omega CDR_{t-1} + \varepsilon_t, \varepsilon_t \sim S_n(0, c_t) \]  
\[(4.6a)\]

\[c_t^{\alpha} = b_1 + b_2 c_{t-1} + b_3 |\varepsilon_{t-1}|^{\alpha} \]  
\[(4.6b)\]

<table>
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<td>(\mu)</td>
<td>0.678 (0.089)</td>
<td>0.684 (0.071)</td>
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<td>(\phi_1)</td>
<td>0.583 (0.378)</td>
<td>0.648 (0.213)</td>
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<td>(\phi_2)</td>
<td>0.196 (0.139)</td>
<td>0.168 (0.107)</td>
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<td>(b_1)</td>
<td>9.8e-09 (1.2e-04)</td>
<td>9.8e-09 (9.7e-07)</td>
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<tr>
<td>(b_2)</td>
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<td>0.928 (0.030)</td>
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<tr>
<td>(b_3)</td>
<td>0.039 (0.022)</td>
<td>0.030 (0.016)</td>
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<td>(c_0)</td>
<td>1.078 (0.277)</td>
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</table>

Standard errors appear in parentheses.

* Ten percent critical value for a \(\chi^2_1\) equals 2.706.

** Five percent critical value for a \(\chi^2_1\) equals 3.841, and 2.5 percent critical value equals 5.024.

Table F.4: Non-linear ARFIMA(2,d,0)-GARCH(1,1) model
Fig.F.1 U.S. real GNP – A. log level, B. growth rate
Fig.F.2 Current depth of recession - CDR
Fig.F.3 Persistence factor at 10 years
Fig. F.4 Impulse responses -- A. negative innovation of 1.5%,
B. positive innovation of 1.5%
(Fig. 4 contd.)

B

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LIST OF REFERENCES


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