12.2 Explain why each of the following sets is not a subsequence of
\[ \{1 \ 3 \ 1 \ 3 \ 1 \ 3 \ 1 \ 3 \ 1 \cdot \cdot \cdot \} : \]
\[ \{1 \ 3 \ 1 \ 3 \ 1 \ 3 \ 1 \cdot \cdot \cdot \} : \]
\[ a) \ \{1 \ 3 \ 1 \ 3 \ 1 \ 3 \ 1 \cdot \cdot \cdot \} , \quad b) \ \{3 \ 3 \ 3 \} , \quad c) \ \{1 \ 2 \ 1 \ 1 \ 1 \ 1 \cdot \cdot \cdot \} . \]

The original sequence has a numerator alternating between 1 and 3, while the denominator starts at 1 and increases by 1 at every odd term. Sequence (a) is not a subsequence because does not follow the same order as the original sequence where \( \frac{1}{2} \) precedes \( \frac{3}{2} \). Alternative (b) is not a subsequence because it is not an infinite sequence. Sequence (c) is not a subsequence because it includes a term, \( \frac{2}{1} \), that is not in the original sequence.

12.7 Suppose that \( \{x_n\}_{n=1}^{\infty} \) is a sequence of real numbers that convergest to \( x_0 \) and that all \( x_n \) and \( x_0 \) are nonzero.

a) Prove that there is a positive number \( B \) such that \( |x_n| \geq B \) for all \( n \).

b) Using a, prove that \( \{1/x_n\} \) converges to \( 1/x_0 \).

a) Since \( x_n \to x_0 \), we may choose \( N \) so that \( |x_n - x_0| < |x_0|/2 \) for \( n \geq N \). Then \( |x_0| \leq |x_n| + |x_0 - x_n| \leq |x_n| + |x_0|/2 \). Subtracting \( |x_0|/2 \) from both ends yields \( |x_n| > |x_0|/2 \) for \( n \geq N \). Now let \( B = \min \{ |x_1|/2, |x_2|/2, \ldots, |x_{N-1}|/2, |x_N|/2 \} \). Since each \( x_n \) and \( x_0 \) are nonzero, \( B > 0 \). For \( n < N, |x_n| \geq 2B > B \) by construction while for \( n \geq N, |x_n| > |x_0|/2 \geq B \) by our original choice of \( N \). This establishes the result.

b) Let \( \varepsilon > 0 \) be arbitrary. Choose \( N \) such that \( |x_n - x_0| < B|x_0|\varepsilon \). Then, for \( n \geq N, \)
\[
|x_n - x_0| = \frac{1}{|x_n|} |x_n - x_0| \leq \frac{1}{|x_n|} B|x_0|\varepsilon < \frac{B}{|x_n|} \varepsilon < \varepsilon
\]
where the last inequality follows from part (a). This establishes that \( 1/x_n \to 1/x_0 \).

12.14 Prove that the strictly positive orthant \( \mathbb{R}_{++}^m \equiv \{(x_1, \ldots, x_m) : x_i > 0 \text{ for } i = 1, \ldots, m\} \) is an open subset of \( \mathbb{R}_{++}^m \) by finding a formula for \( \varepsilon \) in terms of the \( x_i \)'s.

Let \( x \in \mathbb{R}_{++}^m \). Let \( \varepsilon = \min \{x_1, \ldots, x_m\} \). Let \( y \in B_\varepsilon(x) \). Then \( |y_i - x_i| < |y - x| < \varepsilon \). Thus \( -\varepsilon < y_i - x_i < \varepsilon \), which implies \( x_i - \varepsilon < y_i \). But \( x_i - \varepsilon \geq 0 \), so \( y_i > 0 \). Since this holds for every \( i = 1, \ldots, m, y \in \mathbb{R}_{++}^m \).

12.18 Show that closed intervals in \( \mathbb{R}^1 \) — sets of the form \( \{x : a \leq x \leq b\} \) for fixed numbers \( a \) and \( b \) — are closed sets.

Let \([a, b]\) be a closed interval and \( x_n \in [a, b] \) with \( x_n \to x \). We must show \( x \in [a, b] \). By Theorem 12.4, \( a \leq x \leq b \), so \( x \in [a, b] \). Therefore \([a, b]\) is closed.
12.21 For each of the following subsets of the plane, draw the set, state whether it is open, closed, or neither, and justify your answer in a word or two:

\[ a) \{ (x, y) : -1 < x < +1, y = 0 \}, \quad b) \{ (x, y) : x \text{ and } y \text{ are integers} \}, \]
\[ c) \{ (x, y) : x + y = 1 \}, \quad d) \{ (x, y) : x + y < 1 \}, \quad e) \{ (x, y) : x = 0 \text{ or } y = 0 \}. \]

\[ \]

\[ a) \text{This set is neither open (} B_c(0,0) \text{ pokes out}, \text{ nor closed (the limit points } -1,0 \text{ and } 1,0 \text{ are not included).} \]
\[ b) \text{This set is closed. Any convergent sequence with integral coordinates must eventually be constant since there is only one integer point within any distance } \epsilon < 1/2. \]
\[ c) \text{This set is not open (} B_c(1,0) \text{ pokes out). It is closed (if } f(x, y) = x + y, \text{ } f \text{ is continuous and the set is } f^{-1}(\{1\}).. \]
\[ d) \text{This set is open (using the previous } f, \text{ it is } f^{-1}(-\infty,0). \text{ It is not closed since } (1,-1/n) \text{ is in the set and converges to a point outside the set.} \]
\[ e) \text{This set is not open (} B_c(0,0) \text{ pokes out), but is closed as the union of two closed sets (the coordinate axes).} \]