

# *Regularity Criteria for the Three-dimensional Navier–Stokes Equations*

CHONGSHENG CAO & EDRISS S. TITI

ABSTRACT. In this paper we consider the three-dimensional Navier–Stokes equations subject to periodic boundary conditions or in the whole space. We provide sufficient conditions, in terms of one component of the velocity field, or alternatively in terms of one component of the pressure gradient, for the regularity of strong solutions to the three-dimensional Navier–Stokes equations.

*In honor of Professor Ciprian Foias in his 75th birthday*

## 1. INTRODUCTION

The three-dimensional Navier–Stokes equations (NSE) of viscous incompressible fluid read:

$$(1.1) \quad \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = 0,$$

$$(1.2) \quad \nabla \cdot u = 0,$$

$$(1.3) \quad u(x_1, x_2, x_3, 0) = u_0(x_1, x_2, x_3),$$

where  $u = (u_1, u_2, u_3)$ , the velocity field, and  $p$ , the pressure, are the unknowns, and  $\nu > 0$ , the viscosity, is given. We set  $\nabla_h = (\partial_{x_1}, \partial_{x_2})$  to be the horizontal gradient operator and  $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$  the horizontal Laplacian, while  $\nabla$  and  $\Delta$  are the usual gradient and the Laplacian operators, respectively. We equip the system

(1.1)–(1.3) with periodic boundary conditions with period 1. Namely,

$$(1.4) \quad u(x_1+1, x_2, x_3, t) = u(x_1, x_2+1, x_3, t) = u(x_1, x_2, x_3+1, t) \\ = u(x_1, x_2, x_3, t).$$

Because of the periodic boundary conditions we consider here the base domain  $\Omega = (0, 1)^3$ . We emphasize, however, that one can apply our proof almost line by line to obtain same results for the three-dimensional Navier–Stokes equations in the whole space  $\mathbb{R}^3$ .

The question of global regularity for the 3D Navier–Stokes equations is a major open problem in applied analysis. Over the years there has been an intensive work by many authors attacking this problem (see, e.g., [11], [12], [14], [25], [28], [30], [31], [32], [40], [43], [44], [45] and references therein). It is well known that the 2D Navier–Stokes equations have a unique weak and strong solutions which exist globally in time (cf., for example, [12], [25], [40], [43], [44]). In the 3D case, the weak solutions are known to exist globally in time. But, the uniqueness, regularity, and continuous dependence on initial data for weak solutions are still open problems. Furthermore, strong solutions in the 3D case are known to exist for a short interval of time whose length depends on the physical data of the initial–boundary value problem. Moreover, this strong solution is known to be unique (cf., for example, [12], [25], [40], [43]).

Starting from the pioneer works of Prodi [36] and of Serrin [39], many articles were dedicated for providing sufficient conditions for the global regularity of the 3D Navier–Stokes equations (for details see, for example, the survey papers [28], [45] and references therein). Most recently, there has been some progress along these lines (see, for example, [2], [3], [15], [18], [19], [21], [41], [42], and references therein) which states, roughly speaking, that a strong solution  $u$  exists on the time interval  $[0, T]$  for as long as

$$(1.5) \quad u \in L^p([0, T], L^q(\Omega)), \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = 1, \quad \text{for} \quad q \geq 3.$$

Moreover, that have also been some articles dedicated to the study the global regularity of the 3D Navier–Stokes equation by providing some sufficient conditions on the pressure (cf. e.g., [3], [10], [22], [38], [48]). In addition, some other sufficient regularity conditions were established in terms of only one component of the velocity field of the 3D NSE on the whole space  $\mathbb{R}^3$  or under periodic boundary conditions (cf. e.g., [20], [23], [35], [46]). In [7] we introduced a sufficient regularity condition on one component of the velocity field of the 3D Navier–Stokes equations under Dirichlet boundary conditions (see also, [34]).

We denote by  $L^q(\Omega)$  and  $H^m(\Omega)$  the usual  $L^q$ –Lebesgue and Sobolev spaces, respectively (cf. [1]). We denote by

$$(1.6) \quad \|\phi\|_q = \left( \int_{\Omega} |\phi(x)|^q dx_1 dx_2 dx_3 \right)^{1/q}, \quad \text{for every } \phi \in L^q(\Omega).$$



where  $u = (u_1, u_2, u_3)$  is a weak solution with the initial datum  $u_0$ , then  $u$  is a strong solution of the 3D Navier–Stokes equations which exists on the interval  $[0, T]$ . Moreover,  $u$  is the only weak and strong solution on the interval  $[0, T]$  with the initial datum  $u_0$ . In particular, if (1.9) holds for all  $T > 0$ , then there is a unique global (in time) strong solution for the 3D NSE with the initial datum  $u_0$ .

As a corollary of this result we provide another criterion for regularity of the 3D NSE in terms of one component of the pressure gradient,  $\partial_{x_3} p$ . Specifically, we show that the condition:

$$\partial_{x_3} p \in L^\mu([0, T], L^\sigma(\Omega)); \quad \text{with } \sigma > \frac{21}{16}, \mu \geq 1 \text{ and } \frac{3}{\sigma} + \frac{2}{\mu} < \frac{20}{7},$$

implies (1.9), and hence the corresponding three-dimensional regularity result.

We remark that a slightly weaker conditional regularity result for the 3D NSE which is involving conditions on one component of the pressure gradient, under different type of boundary conditions, has been reported in [9]. Due to the difference in the boundary conditions we observe that the tools used in [9] are based on the techniques developed in [8] and are completely different than the ones presented here.

For convenience, we recall the following three-dimensional Sobolev and Ladyzhenskaya inequalities (see, e.g., [1], [12], [17], [27]). There exists a positive constant  $C_r$  such that

$$\begin{aligned} (1.10) \quad \|\psi\|_r &\leq C_r \|\psi\|_2^{(6-r)/(2r)} (\|\partial_{x_1} \psi\|_2 + \|\psi\|_2)^{(r-2)/(2r)} \\ &\quad \times (\|\partial_{x_2} \psi\|_2 + \|\psi\|_2)^{(r-2)/(2r)} (\|\partial_{x_3} \psi\|_2 + \|\psi\|_2)^{(r-2)/(2r)} \\ &\leq C_r \|\psi\|_2^{(6-r)/(2r)} \|\psi\|_{H^1(\Omega)}^{(3(r-2))/(2r)}, \end{aligned}$$

for every  $\psi \in H^1(\Omega)$ ,  $2 \leq r \leq 6$ .

We dedicate this work to Professor Ciprian Foias, a teacher, friend and collaborator, on the occasion of his 75th birthday.

## 2. THE MAIN RESULT

In this section we will prove our main result, which states that the strong solution to system (1.1)–(1.3) exists on the interval  $[0, T]$  provided the assumption (1.9) on  $u_3$  holds.

**Theorem 2.1.** *Let  $u_0 \in V$ , and let  $u = (u_1, u_2, u_3)$  be a Leray–Hopf weak solution to system (1.1)–(1.3) with the initial value  $u_0$ . Let  $T > 0$ , and suppose that*

$u_3$  satisfies the condition (1.9), namely,

$$(2.1) \quad \int_0^T \|u_3(s)\|_\alpha^\beta ds \leq M, \quad \text{with } \frac{7}{2} < \alpha, \quad 1 \leq \beta < \infty, \text{ and} \\ \frac{3}{\alpha} + \frac{2}{\beta} < \frac{2(\alpha + 1)}{3\alpha}, \text{ or}$$

$$(2.2) \quad \sup_{0 \leq t < T} \|u_3(t)\|_\alpha \leq M, \quad \text{with } \alpha > \frac{7}{2},$$

for some  $M > 0$ . Then  $u$  is a strong solution of system (1.1)–(1.3) on the interval  $[0, T]$ . Moreover, it is the only weak solution on  $[0, T]$  with the initial datum  $u_0$ .

*Proof.* By standard procedure for the 3D Navier–Stokes equations with periodic boundary condition (see, e.g., [12], [14], [28], [37], [40], [43], [44]) one can show that there exists a global in time Leray–Hopf weak solution to the system (1.1)–(1.3), if  $u_0 \in H$ . Furthermore, one can show the short time existence of a unique strong solution if  $u_0 \in V$ . In addition, this strong solution is the only weak solution, with the initial datum  $u_0$ , on the maximal interval of existence of the strong solution.

Suppose that  $u$  is the strong solution with initial value  $u_0 \in V$  such that  $u \in C([0, \mathcal{T}^*), V) \cap L^2([0, \mathcal{T}^*), H^2(\Omega))$ , where  $[0, \mathcal{T}^*)$  is the maximal interval of existence of the unique strong solution. If  $\mathcal{T}^* \geq T$ , then there is nothing to prove. If, on the other hand,  $\mathcal{T}^* < T$ , our strategy is to show that the  $H^1$  norm of this strong solution is bounded on  $[0, \mathcal{T}^*)$ , provided condition (1.9) is valid. As a result  $[0, \mathcal{T}^*)$  cannot be a maximal interval of existence and consequently  $\mathcal{T}^* \geq T$ , which will conclude our proof.

From now on we focus on the strong solution,  $u$ , on its maximal interval of existence  $[0, \mathcal{T}^*)$ , where we assume that  $\mathcal{T}^* < T$ . As we have observed earlier the strong solution  $u$  will also be the only weak solution on the interval  $[0, \mathcal{T}^*)$ . Therefore, by the energy inequality (1.8) for Leray–Hopf weak solutions we have (see, for example, [12], [43] or [44] for details)

$$(2.3) \quad \|u(t)\|_2^2 + \nu \int_0^t \|\nabla u(s)\|_2^2 ds \leq K_1,$$

where

$$(2.4) \quad K_1 = \|u_0\|_2^2.$$

**2.1.  $H^1$  estimates** Next, let us show that the  $H^1$  norm of the strong solution  $u$  is bounded on interval  $[0, \mathcal{T}^*)$ . Denote by

$$(2.5) \quad V(t) = \int_0^t \left( 1 + \|\nabla u(s)\|_2^2 + \|u_3(s)\|_\alpha^\beta \right) \|\nabla u(s)\|_2^2 ds,$$

where  $\alpha$  and  $\beta$  are as in (2.1) or in (2.2). Taking the inner product of the equation (1.1) with  $-\Delta_h u$  in  $L^2$ , and using the fact that the Stokes operator is the same as the Laplacian operator under periodic boundary conditions, we obtain

$$(2.6) \quad \frac{1}{2} \frac{d\|\nabla_h u\|_2^2}{dt} + \nu \|\nabla_h \nabla u\|_2^2 = \int_{\Omega} [(u \cdot \nabla)u] \cdot \Delta_h u \, dx_1 \, dx_2 \, dx_3.$$

By integration by parts a few times, we get

$$(2.7) \quad \begin{aligned} & - \int_{\Omega} (u \cdot \nabla)u \cdot \Delta_h u \, dx_1 \, dx_2 \, dx_3 \\ &= - \int_{\Omega} \sum_{k,j=1}^3 \sum_{\ell=1}^2 u_j \frac{\partial u_k}{\partial x_j} \frac{\partial^2 u_k}{\partial x_\ell^2} \, dx_1 \, dx_2 \, dx_3 \\ &= \int_{\Omega} \sum_{k,j=1}^3 \sum_{\ell=1}^2 \frac{\partial u_j}{\partial x_\ell} \frac{\partial u_k}{\partial x_j} \frac{\partial u_k}{\partial x_\ell} \, dx_1 \, dx_2 \, dx_3 + \int_{\Omega} \sum_{k,j=1}^3 \sum_{\ell=1}^2 u_j \frac{\partial^2 u_k}{\partial x_j \partial x_\ell} \frac{\partial u_k}{\partial x_\ell} \, dx_1 \, dx_2 \, dx_3. \end{aligned}$$

Thanks to the incompressibility condition (1.2) we have

$$\int_{\Omega} (u \cdot \nabla)w \cdot w \, dx_1 \, dx_2 \, dx_3 = 0,$$

and consequently we have

$$\int_{\Omega} \sum_{k,j=1}^3 \sum_{\ell=1}^2 u_j \frac{\partial^2 u_k}{\partial x_j \partial x_\ell} \frac{\partial u_k}{\partial x_\ell} \, dx_1 \, dx_2 \, dx_3 = 0.$$

Thus, from (2.7) and the above we have

$$\begin{aligned} & - \int_{\Omega} (u \cdot \nabla)u \cdot \Delta_h u \, dx_1 \, dx_2 \, dx_3 \\ &= \int_{\Omega} \sum_{k,j=1}^3 \sum_{\ell=1}^2 \frac{\partial u_j}{\partial x_\ell} \frac{\partial u_k}{\partial x_j} \frac{\partial u_k}{\partial x_\ell} \, dx_1 \, dx_2 \, dx_3 \\ &= \int_{\Omega} \left\{ \left(\frac{\partial u_1}{\partial x_1}\right)^3 + \left(\frac{\partial u_2}{\partial x_2}\right)^3 + \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right) \left[ \left(\frac{\partial u_1}{\partial x_2}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \right] \right. \\ & \quad \left. + \sum_{k,\ell=1}^2 \frac{\partial u_3}{\partial x_\ell} \frac{\partial u_k}{\partial x_3} \frac{\partial u_k}{\partial x_\ell} + \sum_{j,\ell=1}^2 \frac{\partial u_j}{\partial x_\ell} \frac{\partial u_3}{\partial x_j} \frac{\partial u_3}{\partial x_\ell} + \sum_{\ell=1}^2 \frac{\partial u_3}{\partial x_3} \frac{\partial u_3}{\partial x_\ell} \frac{\partial u_3}{\partial x_\ell} \right\} \, dx_1 \, dx_2 \, dx_3 = \end{aligned}$$

$$\begin{aligned}
 &= \int_{\Omega} \left\{ \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 - \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} + \left( \frac{\partial u_2}{\partial x_2} \right)^2 \right] \right. \\
 &\quad + \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \left[ \left( \frac{\partial u_1}{\partial x_2} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \right] \\
 &\quad \left. + \sum_{k,\ell=1}^2 \frac{\partial u_3}{\partial x_\ell} \frac{\partial u_k}{\partial x_3} \frac{\partial u_k}{\partial x_\ell} + \sum_{k,\ell=1}^2 \frac{\partial u_k}{\partial x_\ell} \frac{\partial u_3}{\partial x_k} \frac{\partial u_3}{\partial x_\ell} + \sum_{\ell=1}^2 \frac{\partial u_3}{\partial x_3} \frac{\partial u_3}{\partial x_\ell} \frac{\partial u_3}{\partial x_\ell} \right\} dx_1 dx_2 dx_3.
 \end{aligned}$$

By substituting the incompressibility condition (1.2) in the above we obtain

$$\begin{aligned}
 & - \int_{\Omega} (u \cdot \nabla) u \cdot \Delta_h u \, dx_1 dx_2 dx_3 \\
 &= \int_{\Omega} \left\{ - \frac{\partial u_3}{\partial x_3} \left[ \left( \frac{\partial u_1}{\partial x_1} \right)^2 + \left( \frac{\partial u_2}{\partial x_2} \right)^2 - \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} + \left( \frac{\partial u_1}{\partial x_2} \right)^2 + \left( \frac{\partial u_2}{\partial x_1} \right)^2 + \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \right] \right. \\
 &\quad \left. + \sum_{\ell=1}^2 \frac{\partial u_3}{\partial x_\ell} \left[ \sum_{k=1}^2 \frac{\partial u_k}{\partial x_3} \frac{\partial u_k}{\partial x_\ell} + \sum_{k=1}^2 \frac{\partial u_3}{\partial x_k} \frac{\partial u_k}{\partial x_\ell} + \frac{\partial u_3}{\partial x_3} \frac{\partial u_3}{\partial x_\ell} \right] \right\} dx_1 dx_2 dx_3.
 \end{aligned}$$

By integration by parts in the above we have

$$\begin{aligned}
 (2.8) \quad & - \int_{\Omega} (u \cdot \nabla) u \cdot \Delta_h u \, dx_1 dx_2 dx_3 \\
 & \leq C \int_{\Omega} |u_3| |\nabla_h u| |\nabla_h \nabla u| \, dx_1 dx_2 dx_3 \\
 & \quad + C \int_{\Omega} |u_3| |\partial u / \partial x_3| |\nabla_h \nabla u| \, dx_1 dx_2 dx_3.
 \end{aligned}$$

By Hölder’s inequality and (1.10) we get

$$\begin{aligned}
 (2.9) \quad & \int_{\Omega} |u_3| |\nabla_h u| |\nabla_h \nabla u| \, dx_1 dx_2 dx_3 \\
 & \leq C \|u_3\|_{\alpha} \|\nabla_h u\|_{(2\alpha)/(\alpha-2)} \|\nabla_h \nabla u\|_2 \\
 & \leq C \|u_3\|_{\alpha} \|\nabla_h u\|_2^{(\alpha-3)/\alpha} \|\nabla_h \nabla u\|_2^{(\alpha+3)/\alpha} + C \|u_3\|_{\alpha} \|\nabla_h u\|_2 \|\nabla_h \nabla u\|_2,
 \end{aligned}$$

and

$$\begin{aligned}
 (2.10) \quad & \int_{\Omega} |u_3| |\partial u / \partial x_3| |\nabla_h \nabla u| \, dx_1 dx_2 dx_3 \\
 & \leq C \|u_3\|_{\alpha} \|\partial u / \partial x_3\|_{(2\alpha)/(\alpha-2)} \|\nabla_h \nabla u\|_2.
 \end{aligned}$$

Notice that by (1.10) we also have

$$\begin{aligned}
(2.11) \quad & \left\| \frac{\partial u}{\partial x_3} \right\|_{(2\alpha)/(\alpha-2)} \\
& \leq C \left\| \frac{\partial u}{\partial x_3} \right\|_2^{(\alpha-3)/\alpha} \left( \left\| \frac{\partial \nabla_h u}{\partial x_3} \right\|_2 + \left\| \frac{\partial u}{\partial x_3} \right\|_2 \right)^{2/\alpha} \left( \left\| \frac{\partial^2 u}{\partial x_3^2} \right\|_2 + \left\| \frac{\partial u}{\partial x_3} \right\|_2 \right)^{1/\alpha} \\
& \leq C \|\nabla u\|_2^{(\alpha-3)/\alpha} \|\nabla_h \nabla u\|_2^{2/\alpha} \|\Delta u\|_2^{1/\alpha} + C \|\nabla u\|_2^{(\alpha-2)/\alpha} \|\nabla_h \nabla u\|_2^{2/\alpha} \\
& \quad + C \|\nabla u\|_2^{(\alpha-1)/\alpha} \|\Delta u\|_2^{1/\alpha} + C \|\nabla u\|_2.
\end{aligned}$$

As a result of the above we obtain

$$\begin{aligned}
(2.12) \quad & \int_{\Omega} |u_3| \left| \frac{\partial u}{\partial x_3} \right| |\nabla_h \nabla u| \, dx_1 \, dx_2 \, dx_3 \\
& \leq C \|u_3\|_{\alpha} \left[ \|\nabla u\|_2^{(\alpha-3)/\alpha} \|\nabla_h \nabla u\|_2^{2/\alpha} \|\Delta u\|_2^{1/\alpha} + \|\nabla u\|_2^{(\alpha-2)/\alpha} \|\nabla_h \nabla u\|_2^{2/\alpha} \right. \\
& \quad \left. + \|\nabla u\|_2^{(\alpha-1)/\alpha} \|\Delta u\|_2^{1/\alpha} + \|\nabla u\|_2 \right] \|\nabla_h \nabla u\|_2.
\end{aligned}$$

Therefore, by (2.9), (2.12), (2.8) and (2.6) we get

$$\begin{aligned}
(2.13) \quad & \frac{d\|\nabla_h u\|_2^2}{dt} + \nu \|\nabla_h \nabla u\|_2^2 \\
& \leq C \|u_3\|_{\alpha} \|\nabla_h u\|_2^{(\alpha-3)/\alpha} \|\nabla_h \nabla u\|_2^{(\alpha+3)/\alpha} + C \|u_3\|_{\alpha} \|\nabla_h u\|_2 \|\nabla_h \nabla u\|_2 \\
& \quad + C \|u_3\|_{\alpha} \left[ \|\nabla u\|_2^{(\alpha-3)/\alpha} \|\nabla_h \nabla u\|_2^{2/\alpha} \|\Delta u\|_2^{1/\alpha} + \|\nabla u\|_2^{(\alpha-2)/\alpha} \|\nabla_h \nabla u\|_2^{2/\alpha} \right. \\
& \quad \left. + \|\nabla u\|_2^{(\alpha-1)/\alpha} \|\Delta u\|_2^{1/\alpha} + \|\nabla u\|_2 \right] \|\nabla_h \nabla u\|_2,
\end{aligned}$$

and by Young's inequality we have

$$\begin{aligned}
& \frac{d\|\nabla_h u\|_2^2}{dt} + \nu \|\nabla_h \nabla u\|_2^2 \\
& \leq C \left( 1 + \|u_3\|_{\alpha}^{(2\alpha)/(\alpha-3)} \right) \|\nabla_h u\|_2^2 \\
& \quad + C \|u_3\|_{\alpha}^{(2/\alpha)/(\alpha-2)} \|\nabla u\|_2^{(2(\alpha-3))/(\alpha-2)} \|\Delta u\|_2^{2/(\alpha-2)} \\
& \quad + C \|u_3\|_{\alpha}^{(2/\alpha)/(\alpha-2)} \|\nabla u\|_2^2 + C \|u_3\|_{\alpha}^2 \|\nabla u\|_2^{(2(\alpha-1))/\alpha} \|\Delta u\|_2^{2/\alpha} + C \|u_3\|_{\alpha}^2 \|\nabla u\|_2^2 \leq
\end{aligned}$$

$$\begin{aligned} &\leq C \left( 1 + \|u_3\|_{\alpha}^{(2\alpha)/(\alpha-3)} \right) \|\nabla u\|_2^2 \\ &\quad + C \|u_3\|_{\alpha}^{(2/\alpha)/(\alpha-2)} \|\nabla u\|_2^{(2(\alpha-3))/(\alpha-2)} \|\Delta u\|_2^{2/(\alpha-2)}. \end{aligned}$$

Thanks to Gronwall’s inequality we obtain

$$(2.14) \quad \|\nabla_h u(t)\|_2^2 - \|\nabla_h u(0)\|_2^2 + \nu \int_0^t \|\nabla_h \nabla u(s)\|_2^2 ds \leq H(t),$$

where

$$\begin{aligned} (2.15) \quad H(t) = & C \int_0^t \left( 1 + \|u_3(s)\|_{\alpha}^{(2\alpha)/(\alpha-3)} \right) \|\nabla u(s)\|_2^2 ds \\ & + C \left( \int_0^t \|u_3(s)\|_{\alpha}^{(2\alpha)/(\alpha-3)} \|\nabla u(s)\|_2^2 ds \right)^{(\alpha-3)/(\alpha-2)} \\ & \times \left( \int_0^t \|\Delta u(s)\|_2^2 ds \right)^{1/(\alpha-2)}. \end{aligned}$$

Taking the inner product of the equation (1.1) with  $-\Delta u$  in  $L^2$ , and using the fact that the Stokes operator is the same as the Laplacian operator under periodic boundary conditions, we obtain

$$\begin{aligned} (2.16) \quad &\frac{1}{2} \frac{d \|\nabla u\|_2^2}{dt} + \nu \|\Delta u\|_2^2 \\ &= \int_{\Omega} [(u \cdot \nabla) u] \cdot \Delta u \, dx_1 \, dx_2 \, dx_3 \\ &\leq \int_{\Omega} \left[ |u| |\nabla_h u| + |u_3| \left| \frac{\partial u}{\partial x_3} \right| \right] |\Delta u| \, dx_1 \, dx_2 \, dx_3 \\ &\leq [\|u\|_6 \|\nabla_h u\|_3 + \|u_3\|_{\alpha} \|\nabla u\|_{(2\alpha)/(\alpha-2)}] \|\Delta u\|_2. \end{aligned}$$

By (1.10) we have

$$\begin{aligned} (2.17) \quad &\|u\|_6 \leq C \left( \left\| \frac{\partial u}{\partial x_3} \right\|_2 + \|u\|_2 \right)^{1/3} (\|\nabla_h u\|_2 + \|u\|_2)^{2/3}; \\ &\|\nabla_h u\|_3 \leq C \|\nabla_h u\|_2^{1/2} \|\nabla \nabla_h u\|_2^{1/2} + C \|\nabla_h u\|_2; \\ &\|\nabla u\|_{(2\alpha)/(\alpha-2)} \leq C \|\nabla u\|_2^{(\alpha-3)/\alpha} \|\Delta u\|_2^{3/\alpha} + C \|\nabla u\|_2. \end{aligned}$$

As a result we get

$$\begin{aligned}
 (2.18) \quad & \frac{1}{2} \frac{d \|\nabla u\|_2^2}{dt} + \nu \|\Delta u\|_2^2 \\
 & \leq C \|\nabla u\|_2^{1/3} \|\nabla_h u\|_2^{7/6} \|\nabla_h \nabla u\|_2^{1/2} \|\Delta u\|_2 \\
 & \quad + C \|u\|_2^{1/3} \|\nabla_h u\|_2^{7/6} \|\nabla_h \nabla u\|_2^{1/2} \|\Delta u\|_2 \\
 & \quad + C \|\nabla u\|_2^{1/3} \|u\|_2^{2/3} \|\nabla_h u\|_2^{1/2} \|\nabla_h \nabla u\|_2^{1/2} \|\Delta u\|_2 \\
 & \quad + C \|u\|_2 \|\nabla_h u\|_2^{1/2} \|\nabla_h \nabla u\|_2^{1/2} \|\Delta u\|_2 \\
 & \quad + C \|\nabla u\|_2^{1/3} \|u\|_2^{2/3} \|\nabla_h u\|_2 \|\Delta u\|_2 + C \|u\|_2 \|\nabla_h u\|_2 \|\Delta u\|_2 \\
 & \quad + C \|\nabla u\|_2^{1/3} \|\nabla_h u\|_2^{5/3} \|\Delta u\|_2 + C \|u\|_2^{1/3} \|\nabla_h u\|_2^{5/3} \|\Delta u\|_2 \\
 & \quad + C \|u_3\|_\alpha \left( \|\nabla u\|_2^{(\alpha-3)/\alpha} \|\Delta u\|_2^{3/\alpha} + C \|\nabla u\|_2 \right) \|\Delta u\|_2.
 \end{aligned}$$

By Young’s inequality we obtain

$$\begin{aligned}
 (2.19) \quad & \frac{d \|\nabla u\|_2^2}{dt} + \nu \|\Delta u\|_2^2 \\
 & \leq C \|\nabla u\|_2^{1/3} \|\nabla_h u\|_2^{7/6} \|\nabla_h \nabla u\|_2^{1/2} \|\Delta u\|_2 \\
 & \quad + C \|u\|_2^{1/3} \|\nabla_h u\|_2^{7/6} \|\nabla_h \nabla u\|_2^{1/2} \|\Delta u\|_2 \\
 & \quad + C \|\nabla u\|_2^{4/3} \|u\|_2^{8/3} \|\nabla_h u\|_2^2 + C \|u\|_2^4 \|\nabla_h u\|_2^2 \\
 & \quad + C \|\nabla u\|_2^{2/3} \|u\|_2^{4/3} \|\nabla_h u\|_2^2 + C \|u\|_2^2 \|\nabla_h u\|_2^2 \\
 & \quad + C \|\nabla u\|_2^{2/3} \|\nabla_h u\|_2^{10/3} + C \|u\|_2^{2/3} \|\nabla_h u\|_2^{10/3} \\
 & \quad + C \|u_3\|_\alpha^{(2\alpha)/(\alpha-3)} \|\nabla u\|_2^2 + C \|u_3\|_\alpha^2 \|\nabla u\|_2^2 \\
 & \leq C \|\nabla u\|_2^{1/3} \|\nabla_h u\|_2^{7/6} \|\nabla_h \nabla u\|_2^{1/2} \|\Delta u\|_2 \\
 & \quad + C \|u\|_2^{1/3} \|\nabla_h u\|_2^{7/6} \|\nabla_h \nabla u\|_2^{1/2} \|\Delta u\|_2 \\
 & \quad + C \left( 1 + \|u\|_2^4 + \|\nabla u\|_2^2 + \|u_3\|_\alpha^{(2\alpha)/(\alpha-3)} \right) \|\nabla u\|_2^2.
 \end{aligned}$$

Thanks to Gronwall’s inequality we obtain

$$(2.20) \quad \|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 ds \leq$$

$$\begin{aligned}
 &\leq C \left( \sup_{0 \leq s \leq t} \|\nabla_h \mathbf{u}(s)\|_2 \right) \left( \int_0^t \|\nabla_h \mathbf{u}\|_2^2 \, ds \right)^{1/12} \left( \int_0^t \|\nabla \mathbf{u}\|_2^2 \, ds \right)^{1/6} \\
 &\quad \times \left( \int_0^t \|\nabla_h \nabla \mathbf{u}(s)\|_2^2 \, ds \right)^{1/4} \left( \int_0^t \|\Delta \mathbf{u}(s)\|_2^2 \, ds \right)^{1/2} \\
 &+ C \left( \sup_{0 \leq s \leq t} \|\mathbf{u}(s)\|_2^{1/3} \right) \left( \sup_{0 \leq s \leq t} \|\nabla_h \mathbf{u}(s)\|_2^{2/3} \right) \left( \int_0^t \|\nabla_h \mathbf{u}\|_2^2 \, ds \right)^{1/4} \\
 &\quad \times \left( \int_0^t \|\nabla_h \nabla \mathbf{u}(s)\|_2^2 \, ds \right)^{1/4} \left( \int_0^t \|\Delta \mathbf{u}(s)\|_2^2 \, ds \right)^{1/2} \\
 &+ C \int_0^t \left( 1 + \|\mathbf{u}(s)\|_2^4 + \|\nabla \mathbf{u}(s)\|_2^2 + \|\mathbf{u}_3(s)\|_\alpha^{(2\alpha)/(\alpha-3)} \right) \|\nabla \mathbf{u}(s)\|_2^2 \, ds.
 \end{aligned}$$

Thanks to (2.3), (2.14), and Young’s inequality, we get

$$\begin{aligned}
 (2.21) \quad &\|\nabla \mathbf{u}(t)\|_2^2 - \|\nabla \mathbf{u}(0)\|_2^2 + \nu \int_0^t \|\Delta \mathbf{u}\|_2^2 \, ds \\
 &\leq C \left[ K_1^{1/4} (1 + H(t))^{3/4} + K_1^{5/12} (1 + H(t))^{7/12} \right] \left( \int_0^t \|\Delta \mathbf{u}(s)\|_2^2 \, ds \right)^{1/2} \\
 &\quad + C \int_0^t \left( 1 + \|\mathbf{u}(s)\|_2^4 + \|\nabla \mathbf{u}(s)\|_2^2 + \|\mathbf{u}_3(s)\|_\alpha^{(2\alpha)/(\alpha-3)} \right) \|\nabla \mathbf{u}(s)\|_2^2 \, ds \\
 &\leq C (1 + H(t))^{3/4} \left( \int_0^t \|\Delta \mathbf{u}(s)\|_2^2 \, ds \right)^{1/2} + C V(t),
 \end{aligned}$$

where  $K_1, V(t)$  and  $H(t)$  are as in (2.4), (2.5) and (2.15), respectively. In the case that  $\mathbf{u}_3$  satisfies (2.1) we have

$$\begin{aligned}
 (2.22) \quad H(t) &\leq C \left( \int_0^t \|\mathbf{u}_3(s)\|_\alpha^\beta \|\nabla \mathbf{u}(s)\|_2^2 \, ds \right)^{(2\alpha)/((\alpha-3)\beta)} \\
 &\quad \times \left( \int_0^t \|\nabla \mathbf{u}(s)\|_2^2 \, ds \right)^{((\alpha-3)\beta-2\alpha)/((\alpha-3)\beta)} \\
 &+ C \left( \int_0^t \|\mathbf{u}_3(s)\|_\alpha^\beta \|\nabla \mathbf{u}(s)\|_2^2 \, ds \right)^{(2\alpha)/((\alpha-2)\beta)} \\
 &\quad \times \left( \int_0^t \|\nabla \mathbf{u}(s)\|_2^2 \, ds \right)^{((\alpha-3)\beta-2\alpha)/((\alpha-2)\beta)} \\
 &\quad \times \left( \int_0^t \|\Delta \mathbf{u}(s)\|_2^2 \, ds \right)^{1/(\alpha-2)} \leq
 \end{aligned}$$

$$\begin{aligned} &\leq CV(t)^{(2\alpha)/((\alpha-3)\beta)} K_1^{((\alpha-3)\beta-2\alpha)/((\alpha-3)\beta)} \\ &\quad + CV(t)^{(2\alpha)/((\alpha-2)\beta)} K_1^{((\alpha-3)\beta-2\alpha)/((\alpha-2)\beta)} \left( \int_0^t \|\Delta u(s)\|_2^2 ds \right)^{1/(\alpha-2)} \\ &\leq CV(t)^{(2\alpha)/((\alpha-3)\beta)} + CV(t)^{(2\alpha)/((\alpha-2)\beta)} \left( \int_0^t \|\Delta u(s)\|_2^2 ds \right)^{1/(\alpha-2)}, \end{aligned}$$

where  $V(t)$  is as in (2.5). Thus, we reach,

$$\begin{aligned} (2.23) \quad &\|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 ds \\ &\leq C \left( 1 + V(t)^{(3\alpha)/(2(\alpha-3)\beta)} + V(t)^{(3\alpha)/(2(\alpha-2)\beta)} \left( \int_0^t \|\Delta u(s)\|_2^2 ds \right)^{3/(4(\alpha-2))} \right) \\ &\quad \times \left( \int_0^t \|\Delta u(s)\|_2^2 ds \right)^{1/2} + CV(t). \end{aligned}$$

By Young’s inequality and the condition  $\beta > (6\alpha)/(2\alpha - 7)$ , we obtain

$$\begin{aligned} (2.24) \quad &\|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 ds \\ &\leq C \left( 1 + V(t)^{(3\alpha)/((\alpha-3)\beta)} + V(t)^{(6\alpha)/((2\alpha-7)\beta)} \right) + CV(t) \leq C(1 + V(t)). \end{aligned}$$

On the other hand, when  $u_3$  satisfies (2.2) we get

$$\begin{aligned} (2.25) \quad &H(t) \leq \\ &\leq C \left( \sup_{0 \leq s \leq t} \|u_3(s)\|_\alpha^{(2\alpha)/(\alpha-3)} \right) \int_0^t \|\nabla u(s)\|_2^2 ds + C \left( \sup_{0 \leq s \leq t} \|u_3(s)\|_\alpha^{(2/\alpha)/(\alpha-2)} \right) \\ &\quad \times \left( \int_0^t \|\nabla u(s)\|_2^2 ds \right)^{(\alpha-3)/(\alpha-2)} \left( \int_0^t \|\Delta u(s)\|_2^2 ds \right)^{1/(\alpha-2)} \\ &\leq CM^{(2\alpha)/(\alpha-3)} K_1 + CM^{(2/\alpha)/(\alpha-2)} K_1^{(\alpha-3)/(\alpha-2)} \left( \int_0^t \|\Delta u(s)\|_2^2 ds \right)^{1/(\alpha-2)} \\ &\leq C + C \left( \int_0^t \|\Delta u(s)\|_2^2 ds \right)^{1/(\alpha-2)}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} (2.26) \quad &\|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 ds \\ &\leq C(1 + V(t)) + C \left( \int_0^t \|\Delta u(s)\|_2^2 ds \right)^{1/2+3/(4(\alpha-2))}. \end{aligned}$$

Thanks to Young’s inequality and since  $\alpha > 7/2$ , we get

$$(2.27) \quad \|\nabla u(t)\|_2^2 - \|\nabla u(0)\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 \, ds \leq C(1 + V(t)).$$

Therefore, by Gronwall’s inequality we obtain

$$(2.28) \quad \|\nabla u(t)\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 \, ds \leq C(1 + \|\nabla u(0)\|_2^2) e^{C(K_1+M)}.$$

for all  $t \in [0, \mathcal{T}^*)$ . Therefore, the  $H^1$  norm of the strong solution  $u$  is bounded on the maximal interval of existence  $[0, \mathcal{T}^*)$ , which leads to a contradiction. This completes the proof of Theorem 2.1.  $\square$

**Corollary 2.2.** *Let  $u_0 \in V$ , and let  $u = (u_1, u_2, u_3)$  be a Leray–Hopf weak solution to system (1.1)–(1.3) with the initial datum  $u_0$ . Let  $T > 0$ , and suppose that  $((\partial p)/(\partial x_3)) \in L^\mu([0, T], L^\sigma(\Omega))$ , where  $\sigma > \frac{21}{16}$ ,  $\mu \geq 1$  and  $3/\sigma + 2/\mu < \frac{20}{7}$ . Then there is a unique strong solution  $u$  of system (1.1)–(1.3) on  $[0, T]$ .*

*Proof.* As in the proof of Theorem 2.1 we observe that  $u$  is the only weak solution, which is also a strong solution, with the initial value  $u_0 \in V$  such that  $u \in C([0, \mathcal{T}^*), V) \cap L^2([0, \mathcal{T}^*), H^2(\Omega))$ , where  $[0, \mathcal{T}^*)$  is the maximal interval of existence of the unique strong solution. If  $\mathcal{T}^* \geq T$  then there is nothing to prove. If, on the other hand,  $\mathcal{T}^* < T$  our strategy is to show that the component  $u_3$  of the velocity field  $u$  satisfies condition (2.2) of Theorem 2.1 on the interval  $[0, \mathcal{T}^*)$ . As a result of Theorem 2.1 the strong solution  $u$  exists on the interval  $[0, \mathcal{T}^*]$ . Hence  $[0, \mathcal{T}^*)$  cannot be a maximal interval of existence for the strong solution, which leads to a contradiction and consequently  $\mathcal{T}^* \geq T$ .

As in the proof of Theorem 2.1 we focus, from now on, on the strong solution,  $u$ , on its maximal interval of existence  $[0, \mathcal{T}^*)$ , where we assume that  $\mathcal{T}^* < T$ .

Let  $r > \frac{7}{4}$ , which will be defined later. Taking the inner product of the equation (1.1) with  $(0, 0, (u_3)^{2r-1})$  in  $L^2(\Omega)$ , we get

$$(2.29) \quad \frac{1}{2r} \frac{d\|u_3\|_{2r}^{2r}}{dt} + \frac{(2r-1)\nu}{r^2} \|\nabla((u_3)^r)\|_2^2 \\ = - \int_\Omega (u \cdot \nabla u_3) (u_3)^{2r-1} \, dx_1 \, dx_2 \, dx_3 - \int_\Omega \frac{\partial p}{\partial x_3} (u_3)^{2r-1} \, dx_1 \, dx_2 \, dx_3.$$

By integration by parts, and the incompressibility condition (1.2) we get

$$(2.30) \quad \int_\Omega (u \cdot \nabla u_3) (u_3)^{2r-1} \, dx_1 \, dx_2 \, dx_3 = 0.$$

Moreover, by Hölder’s inequality and (1.10), we obtain

$$(2.31) \quad \left| \int_\Omega \frac{\partial p}{\partial x_3} (u_3)^{2r-1} \, dx_1 \, dx_2 \, dx_3 \right| \leq \left\| \frac{\partial p}{\partial x_3} \right\|_\sigma \|u_3\|_{(\sigma(2r-1))/(\sigma-1)}^{2r-1}.$$

First, let us assume that  $\sigma > \frac{7}{2}$ . Let  $r = r_* = \sigma/2 > \frac{7}{4}$ . By direct calculation we observe that in this case  $(\sigma(2r_* - 1))/(\sigma - 1) = 2r_*$ . As a result of the above we get

$$(2.32) \quad \frac{d\|u_3\|_{2r_*}}{dt} \leq C \left\| \frac{\partial p}{\partial x_3} \right\|_{\sigma}.$$

Thanks to Gronwall’s inequality, and since  $\mu \geq 1$ , we get

$$(2.33) \quad \begin{aligned} \|u_3(t)\|_{2r_*} &\leq \left( \|u_3(0)\|_{2r_*} + C \int_0^t \left\| \frac{\partial p}{\partial x_3}(s) \right\|_{\sigma} ds \right) \\ &\leq \|u_3(0)\|_{2r_*} + C \left( \int_0^t \left\| \frac{\partial p}{\partial x_3}(s) \right\|_{\sigma}^{\mu} ds \right)^{1/\mu} t^{1-1/\mu}. \end{aligned}$$

Next, let us assume that  $\frac{21}{16} < \sigma \leq \frac{7}{2} < 2r$ . By (1.10), we reach

$$(2.34) \quad \begin{aligned} &\left| \int_{\Omega} \frac{\partial p}{\partial x_3} (u_3)^{2r-1} dx_1 dx_2 dx_3 \right| \\ &\leq \left\| \frac{\partial p}{\partial x_3} \right\|_{\sigma} \|u_3\|_{(\sigma(2r-1))/(\sigma-1)}^{2r-1} \\ &\leq C \left\| \frac{\partial p}{\partial x_3} \right\|_{\sigma} \left( \|u_3\|_{2r}^{(2\sigma(2r-1)-3(2r-\sigma))/(2\sigma)} \|\nabla((u_3)^r)\|_2^{(3(2r-\sigma))/(2r\sigma)} + \|u_3\|_{2r}^{2r} \right). \end{aligned}$$

Notice, that in order to apply (1.10) in the argument above it is necessary to require that  $(\sigma(2r - 1))/(\sigma - 1) \leq 6r$ . The last inequality is true for  $\sigma < \frac{3}{2}$  provided

$$(2.35) \quad r < \frac{\sigma}{6 - 4\sigma},$$

and it is always true, i.e. without any restriction on  $r$ , when  $\sigma \geq \frac{3}{2}$ . Moreover, in this case the condition  $(\sigma(2r - 1))/(\sigma - 1) \leq 6r$  implies that  $(3(2r - \sigma))/(2r\sigma) < 2$ , and therefore under these same conditions as above on  $r$  and  $\sigma$  we can applying Young’s inequality and obtain

$$(2.36) \quad \begin{aligned} &\frac{d\|u_3\|_{2r}^{2r}}{dt} + \nu \|\nabla((u_3)^r)\|_2^2 \\ &\leq C \left\| \frac{\partial p}{\partial x_3} \right\|_{\sigma}^{(4r\sigma)/(4r\sigma-3(2r-\sigma))} \|u_3\|_{2r}^{(2r[2\sigma(2r-1)-3(2r-\sigma)]/(4r\sigma-3(2r-\sigma)))} + \|u_3\|_{2r}^{2r}. \end{aligned}$$

Thus, we have

$$(2.37) \quad \frac{d\|u_3\|_{2r}^{(4r\sigma)/(4r\sigma-3(2r-\sigma))}}{dt} \leq C \left\| \frac{\partial p}{\partial x_3} \right\|_{\sigma}^{(4r\sigma)/(4r\sigma-3(2r-\sigma))} + \|u_3\|_{2r}^{(4r\sigma)/(4r\sigma-3(2r-\sigma))}.$$

Observe that in this case the condition  $2r > \sigma$  implies that

$$\frac{4r\sigma}{4r\sigma - 3(2r - \sigma)} > 1,$$

and therefore thanks to Gronwall's inequality, we get

$$(2.38) \quad \|u_3(t)\|_{2r}^{(4r\sigma)/(4r\sigma-3(2r-\sigma))} \leq e^t \left( \|u_3(0)\|_{2r}^{(4r\sigma)/(4r\sigma-3(2r-\sigma))} + C \int_0^t \left\| \frac{\partial p}{\partial x_3}(s) \right\|_{\sigma}^{(4r\sigma)/(4r\sigma-3(2r-\sigma))} ds \right).$$

Thus,

$$(2.39) \quad \|u_3(t)\|_{2r} \leq e^t \left( \|u_3(0)\|_{2r}^{(4r\sigma)/(4r\sigma-3(2r-\sigma))} + C \int_0^t \left\| \frac{\partial p}{\partial x_3}(s) \right\|_{\sigma}^{(4r\sigma)/(4r\sigma-3(2r-\sigma))} ds \right)^{(4r\sigma-3(2r-\sigma))/(4r\sigma)} \leq e^t \left( \|u_3(0)\|_{2r} + C \left( 1 + \int_0^t \left( 1 + \left\| \frac{\partial p}{\partial x_3}(s) \right\|_{\sigma} \right)^{\mu} ds \right) \right),$$

where the last inequality is true for as long as we can find  $r$  which, in addition to all the conditions above, it also satisfies

$$\mu \geq \frac{4r\sigma}{4r\sigma - 3(2r - \sigma)}.$$

Notice that this is also true for as long as

$$(2.40) \quad r \leq \frac{3\sigma\mu}{6\mu + 4\sigma - 4\sigma\mu} = \frac{3}{2\left(\frac{3}{\sigma} + \frac{2}{\mu} - 2\right)}.$$

Summing up, we set:

$$(2.41) \quad r_* = \begin{cases} \frac{\sigma}{2} & \text{when } \sigma > \frac{7}{2}, \mu \geq 1 \text{ and } \frac{3}{\sigma} + \frac{2}{\mu} < \frac{20}{7}, \\ \min \left\{ 2, \frac{\sigma}{4(3-2\sigma)} + \frac{7}{8}, \frac{3}{2 \left( \frac{3}{\sigma} + \frac{2}{\mu} - 2 \right)} \right\} & \\ \text{when } \frac{21}{16} < \sigma \leq \frac{7}{2}, \mu \geq 1 \text{ and } \frac{3}{\sigma} + \frac{2}{\mu} < \frac{20}{7}. & \end{cases}$$

It is then easy to check that  $r_* > \frac{7}{4}$  under the above assumptions. Moreover, the conditions (2.35) and (2.40) both hold for  $r = r_*$ . Therefore, by (2.33) and (2.39) we have in fact shown that  $u_3 \in L^\infty([0, \mathcal{T}^*], L^{2r_*}(\Omega))$ , which implies condition (2.2) for  $\alpha = 2r_*$ , over the interval  $[0, \mathcal{T}^*]$ . As a consequence of the Theorem 2.1,  $u$  is a strong solution on interval  $[0, \mathcal{T}^*]$ , hence  $[0, \mathcal{T}^*)$  cannot be a maximal interval of existence of the strong solution, which leads to a contradiction. This completes the proof of Corollary 2.2.  $\square$

#### ACKNOWLEDGEMENTS

Edriss S. Titi would like to thank the University of Paris 13, where part of this work was completed, for its kind hospitality. This work was supported in part by NSF grants No. DMS-0709228 and No. DMS-0708832, ISF grant No. 120/6, and the Florida International University Foundation.

#### REFERENCES

- [1] R.A. ADAMS, *Sobolev Spaces*, Academic Press, New York, 1975.
- [2] H. BEIRÃO DA VEIGA, *A sufficient condition on the pressure for the regularity of weak solutions to the Navier–Stokes equations*, J. Math. Fluid Mech. **2** (2000), 99–106.
- [3] L.C. BERSELLI, *On a regularity criterion for the solutions to the 3D Navier–Stokes equations*, Differential Integral Equations **15** (2002), 1129–1137.
- [4] L.C. BERSELLI and G.P. GALDI, *Regularity criteria involving the pressure for the weak solutions to the Navier–Stokes equations*, Proc. Amer. Math. Soc. **130** (2002), 3585–3595.
- [5] V.G. BONDAREVSKY, *On the problem of global regularity for 3-dimensional Navier–Stokes equations*, C.R. Math. Rep. Acad. Sci. Canada. **17** (1995), 109–114.
- [6] ———, *A method of finding large sets of data generating global solutions to nonlinear equations: application to the Navier–Stokes equations*, C.R. Acad. Sci. Paris Sr. I Math. **322** (1996), 333–338.
- [7] M. CANNONE, *Harmonic analysis tools for solving the incompressible Navier–Stokes equations*, Handbook of Mathematical Fluid Dynamics, vol. III, North-Holland, Amsterdam, 2004, pp. 161–244.

- [8] C. CAO, J. QIN, and E.S. TITI, *Regularity criterion of solutions for three-dimensional turbulent Channel flows*, Communications in Partial Differential Equations **33**(1-3) (2008), 419–428.
- [9] C. CAO and E.S. TITI, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Ann. Math. **166** (2007), 245–267.
- [10] ———, *Pressure regularity criterion for the three dimensional Navier-Stokes equations in infinite channel* (2007).
- [11] D. CHAE and J. LEE, *Regularity criterion in terms of pressure for the Navier-Stokes equations*, Nonlinear Anal. **46** (2001), 727–735.
- [12] P. CONSTANTIN, *A few results and open problems regarding incompressible fluids*, Notices Amer. Math. Soc. **42** (1995), 658–663.
- [13] P. CONSTANTIN and C. FOIAS, *Navier-Stokes Equations*, The University of Chicago Press, Chicago, IL, 1988.
- [14] C. DOERING and J. GIBBON, *Applied Analysis of the Navier-Stokes Equations*, Cambridge University Press, Cambridge, UK, 1995.
- [15] L. ESCAURIAZA, G.A. SEREGIN, and V. SVERAK,  *$L^{3,\infty}$ -solutions of the Navier-Stokes equations and backward uniqueness*, Russ. Math. Surv. **58** (2003), 211–250.
- [16] H. FUJITA and T. KATO, *On the Navier-Stokes initial value problem, I*, Arch. Rat. Mech. Anal. **3** (1964), 269–315.
- [17] G.P. GALDI, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations*, vol. I, II, Springer-Verlag, New York, 1994.
- [18] Y. GIGA, *Solutions for semilinear parabolic equations in  $L^p$  and regularity of weak solutions of the Navier–Stokes system*, J. Differential Equations **62** (1986), 186–212.
- [19] Y. GIGA and T. MIYAKAWA, *Solutions in  $L_r$  of the Navier–Stokes initial value problem*, Arch. Rational Mech. Anal. **89** (1985), 267–281.
- [20] C. HE, *New sufficient conditions for regularity of solutions to the Navier–Stokes equations*, Adv. Math. Sci. Appl. **12** (2002), 535–548.
- [21] T. KATO, *Strong  $L^p$  solutions of the Navier–Stokes equation in  $R^m$ , with applications to weak solutions*, Math. Z. **187** (1984), 471–480.
- [22] I. KUKAVICA, *Role of the pressure for validity of the energy equality for solutions of the Navier-Stokes equation*, J. Dynamics and Differential Equations **18** (2006), 461–482.
- [23] I. KUKAVICA and M. ZIANE, *One component regularity for the Navier–Stokes equation*, Nonlinearity **19**(2) (2006), 453–469.
- [24] ———, *Regularity of the Navier–Stokes equation in a thin periodic domain with large data*, Discrete and Continuous Dynamical Systems (B) **16** (2006), 67–86.
- [25] O.A. LADYZHENSKAYA, *Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, 1969, English translation, 2nd ed.
- [26] ———, *Unique solvability in the large of three dimensional Cauchy problem for the Navier–Stokes equations in the presence of axial symmetry*, Boundary Value Problems of Mathematical Physics and Related Aspects of Function Theory, Part 2, Edited by O. A. Ladyzhenskaya, Seminar in Mathematics, V.A. Steklov Mathematical Institute, Leningrad, vol. 7, 1970, pp. 70–79.
- [27] ———, *The Boundary Value Problems of Mathematical Physics* (1985).
- [28] ———, *The sixth millennium problem: Navier–Stokes equations, existence and smoothness*, Uspekhi Mat. Nauk **58**(2) (2003), 45–78, English translation in Russian Math. Surveys, 58(2) (2003), 251–286. (Russian)

- [29] S. LEIBOVICH, A. MAHALOV, and E.S. TITI, *Invariant helical subspaces for the Navier–Stokes equations*, Arch. Rational Mech. Anal. **112** (1990), 193–222.
- [30] P.G. LEMARIÉ–RIEUSSET, *Recent Developments in the Navier–Stokes Problem*, Chapman & Hall, London, 2002.
- [31] J.L. LIONS, *Quelques Méthodes De Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.
- [32] P.L. LIONS, *Mathematical Topics in Fluid Mechanics: Volume 1: Incompressible Models*, Oxford University Press, 1996.
- [33] Y. MEYER, *Oscillating patterns in some nonlinear evolution equations*, Mathematical Foundation of Turbulent Viscous Flows, Lecture Notes in Math., vol. 1871, Springer, Berlin, 2006, pp. 101–187.
- [34] I. MOISE, *Regularity of Navier–Stokes equations*, Differential Integral Equations **19** (2006), 31–50.
- [35] M. POKORNÝ, *On the result of He concerning the smoothness of solutions to the Navier–Stokes equations*, Electron. J. Differential Equations **11** (2003), 1–8.
- [36] G. PRODI, *Un teorema di unicità per le equazioni di Navier–Stokes*, Ann. Mat. Pura Appl. **48** (1959), 173–182.
- [37] G. RAUGEL and G.R. SELL, *Navier–Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions*, J. Amer. Math. Soc. **6** (1993), 503–568.
- [38] G. SEREGIN and V. SVERÁK, *Navier–Stokes equations with lower bounds on the pressure*, Arch. Ration. Mech. Anal. **163**(1) (2002), 65–86.
- [39] J. SERRIN, *On the interior regularity of weak solutions of the Navier–Stokes equations*, Arch. Rational Mech. Anal. **9** (1962), 187–191.
- [40] H. SOHR, *The Navier–Stokes Equations. An Elementary Functional Analytic Approach*, Birkhäuser Verlag, Basel, 2001.
- [41] ———, *A regularity class for the Navier–Stokes equations in Lorentz spaces*, J. Evol. Equ. **1** (2001), 441–467.
- [42] ———, *A generalization of Serrin’s regularity criterion for the Navier–Stokes equations*, Quad. Mat. **10** (2002), 321–347.
- [43] R. TEMAM, *Navier–Stokes Equations, Theory and Numerical Analysis*, North-Holland, Amsterdam, 1984.
- [44] ———, *Navier–Stokes Equations and Nonlinear Functional Analysis*, 2nd, SIAM, 1995.
- [45] ———, *Some developments on Navier–Stokes equations in the second half of the 20th century*, Development of Mathematics 1950—2000, Birkhauser, Basel, 2000, pp. 1049–1106.
- [46] Y. ZHOU, *A new regularity criterion for the Navier–Stokes equations in terms of the gradient of one velocity component*, Methods Appl. Anal. **9** (2002), 563–578.
- [47] ———, *Regularity criteria in terms of pressure for the 3–D Navier–Stokes equations in a generic domain*, Math. Ann. **328** (2004), 173–192.
- [48] ———, *On regularity criteria in terms of pressure for the Navier–Stokes equations in  $\mathbb{R}^3$* , Proc. Amer. Math. Soc. **134** (2005), 149–156.

C. CAO:

Department of Mathematics  
Florida International University  
University Park  
Miami, FL 33199, U.S.A.  
E-MAIL: [caoc@fiu.edu](mailto:caoc@fiu.edu)

E. S. TITI:

Department of Mathematics  
and Department of Mechanical and Aerospace Engineering  
University of California  
Irvine, CA 92697-3875, U.S.A.

*Also:*

Department of Computer Science and Applied Mathematics  
Weizmann Institute of Science  
Rehovot 76100, Israel  
E-MAIL: [etiti@math.uci.edu](mailto:etiti@math.uci.edu), [edriiss.titi@weizmann.ac.il](mailto:edriiss.titi@weizmann.ac.il)

KEY WORDS AND PHRASES:

Three-dimensional Navier–Stokes equations; regularity criterion; global regularity.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 35Q35, 65M70.

*Received: Submitted: June 26, 2008; revised: July 7, 2008.*

*Article electronically published on October 6th, 2008.*