

**GLOBAL REGULARITY CRITERION FOR THE 3D NAVIER–STOKES EQUATIONS
INVOLVING ONE ENTRY OF THE VELOCITY GRADIENT TENSOR**

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ABSTRACT. In this paper we provide a sufficient condition, in terms of only one of the nine entries of the gradient tensor, i.e., the Jacobian matrix of the velocity vector field, for the global regularity of strong solutions to the three-dimensional Navier–Stokes equations in the whole space, as well as for the case of periodic boundary conditions.

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1. INTRODUCTION

The three-dimensional Navier–Stokes equations (NSE) of viscous incompressible fluid read:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad (1)$$

$$\nabla \cdot u = 0, \quad (2)$$

$$u(x_1, x_2, x_3, 0) = u_0(x_1, x_2, x_3), \quad (3)$$

where $u = (u_1, u_2, u_3)$, the velocity field, and p , the pressure, are the unknowns, and $\nu > 0$, the viscosity, is given. We set $\nabla_h = (\partial_{x_1}, \partial_{x_2})$ to be the horizontal gradient operator and $\Delta_h = \partial_{x_1}^2 + \partial_{x_2}^2$ the horizontal Laplacian, while ∇ and Δ are the usual gradient and the Laplacian operators, respectively. In this paper we consider finite energy solutions of the system (1)–(3) in the whole space \mathbb{R}^3 , that decay at infinity. However, we remark that one can apply our proof, nearly line by line, to establish same result for the three-dimensional Navier–Stokes equations in a periodic domain.

The question of global regularity for the 3D Navier–Stokes equations is a major challenging problem in applied analysis. Over the years there has been an intensive work by many authors attacking this problem (see, e.g., [6], [7], [9], [19], [21], [22], [24], [25], [32], [35], [36], [37] and references therein). It is well-known that the 2D Navier–Stokes equations have a unique weak and strong solutions which exist globally in time (cf., for example, [7], [19], [32], [35], [36]). In the 3D case, the weak solutions are known to exist globally in time. But, the uniqueness, regularity, and continuous dependence on initial data for weak solutions are still open problems. Furthermore, strong solutions in the 3D case are known to exist for a short interval of time whose length depends on the physical data of the initial–boundary value problem. Moreover, this strong solution is known to be unique and depend continuously on the initial data (cf., for example, [7], [19], [32], [35]).

Starting from the pioneer works of Prodi [28] and of Serrin [31], many articles were dedicated for providing sufficient conditions for the global regularity of the 3D Navier–Stokes equations (for details see, for example, the survey papers [21], [37] and references therein). Most recently, there has been some progress along these lines (see, for example, [2], [3], [10], [11], [13], [14], [16], [33], [34], and references therein) which states, roughly speaking,

that a strong solution u exists on the time interval $[0, T]$ for as long as

$$u \in L^p([0, T], L^q), \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} = 1, \quad \text{for} \quad q \geq 3. \quad (4)$$

Moreover, that has also been some works dedicated to the study the global regularity of the 3D Navier–Stokes equations by providing some sufficient conditions on the pressure (cf. e.g., [3], [4], [5], [8], [17], [30], [39]). In addition, some other sufficient regularity conditions were established in terms of only one component of the velocity field of the 3D NSE on the whole space \mathbb{R}^3 or under periodic boundary conditions (cf. e.g., [4], [15], [18], [26], [27], [38]).

We denote by L^q and H^m the usual L^q –Lebesgue and Sobolev spaces, respectively (cf. [1]), and by

$$\|\phi\|_q = \left(\int_{\mathbb{R}^3} |\phi(x)|^q dx_1 dx_2 dx_3 \right)^{\frac{1}{q}}, \quad \text{for every } \phi \in L^q. \quad (5)$$

We set

$$\mathcal{V} = \{ \phi : \text{the three-dimensional vector valued } C_0^\infty \text{ functions and } \nabla \cdot \phi = 0 \},$$

which will form the space of test functions. Let H and V be the closure spaces of \mathcal{V} in L^2 under L^2 –topology, and in H^1 under H^1 –topology, respectively. Let $u_0 \in H$, we say u is a Leray–Hopf weak solution to the system (1)–(3) on the interval $[0, T]$ with initial value u_0 if u satisfies the following three conditions:

- (1) $u \in C_w([0, T], H) \cap L^2([0, T], V)$, and $\partial_t u \in L^1([0, T], V')$, where V' is the dual space of V ;
- (2) the weak formulation of the NSE:

$$\begin{aligned} & \int_{\mathbb{R}^3} u(x, t) \cdot \phi(x, t) dx - \int_{\mathbb{R}^3} u(x, t_0) \cdot \phi(x, t_0) dx \\ &= \int_{t_0}^t \int_{\mathbb{R}^3} [u(x, s) \cdot (\phi_t(x, s) + \nu \Delta \phi(x, s))] dx ds \\ &+ \int_{t_0}^t \int_{\mathbb{R}^3} [(u(x, s) \cdot \nabla) \phi(x, s)] \cdot u(x, s) dx, \end{aligned}$$

for every test function $\phi \in C^\infty([0, T], \mathcal{V})$, and for almost every $t, t_0 \in [0, T]$;

- (3) the energy inequality:

$$\|u(t)\|_2^2 + \nu \int_{t_0}^t \|\nabla u(s)\|_2^2 ds \leq \|u(t_0)\|_2^2, \quad (6)$$

for every t and almost every t_0 .

Moreover, if $u_0 \in V$, a weak solution is called strong solution of (1)–(3) on $[0, T]$ if, in addition, it satisfies

$$u \in C([0, T], V) \cap L^2([0, T], H^2), \quad \text{and} \quad \partial_t u \in L^2([0, T], H).$$

In this case, one also has energy equality in (6) instead of inequality, and the equality holds for every t_0 .

In this paper, we provide sufficient conditions, in terms of only one of the nine components of the gradient of velocity field, i.e., the velocity Jacobian matrix, that guarantee the global regularity of the 3D NSE. Specifically, if $u_0 \in V$, and if for some $T > 0$ and some k, j , with $1 \leq k, j \leq 3$, we have

$$\frac{\partial u_j}{\partial x_k} \in L^\beta([0, T], L^\alpha(\mathbb{R}^3)); \quad \text{when } k \neq j, \text{ and where } \alpha > 3, 1 \leq \beta < \infty, \text{ and } \frac{3}{\alpha} + \frac{2}{\beta} < \frac{\alpha + 3}{2\alpha}, \quad (7)$$

or

$$\frac{\partial u_j}{\partial x_j} \in L^\beta([0, T], L^\alpha(\mathbb{R}^3)); \quad \text{where } \alpha > 2, 1 \leq \beta < \infty, \text{ and } \frac{3}{\alpha} + \frac{2}{\beta} < \frac{3(\alpha + 2)}{4\alpha}, \quad (8)$$

where $u = (u_1, u_2, u_3)$ is a weak solution with the initial datum u_0 on $[0, T]$, then u is a strong solution of the 3D Navier–Stokes equations which exists on the interval $[0, T]$. Moreover, u is the only weak and strong solution on the interval $[0, T]$ with the initial datum u_0 . In particular, if (7) or (8) holds for all $T > 0$, then there is a unique global (in time) strong solution for the 3D NSE with the initial datum u_0 .

For convenience, we recall the following version of the three-dimensional Sobolev and Ladyzhenskaya inequalities in the whole space \mathbb{R}^3 (see, e.g., [1], [7], [12], [20]). There exists a positive constant C_r such that

$$\begin{aligned} \|\psi\|_r &\leq C_r \|\psi\|_2^{\frac{6-r}{2r}} \|\partial_{x_1}\psi\|_2^{\frac{r-2}{2r}} \|\partial_{x_2}\psi\|_2^{\frac{r-2}{2r}} \|\partial_{x_3}\psi\|_2^{\frac{r-2}{2r}} \\ &\leq C_r \|\psi\|_2^{\frac{6-r}{2r}} \|\psi\|_{H^1(\mathbb{R}^3)}^{\frac{3(r-2)}{2r}}, \end{aligned} \quad (9)$$

for every $\psi \in H^1(\mathbb{R}^3)$ and every $r \in [2, 6]$. Observe that in case of periodic boundary conditions, one would have instead of (9) the following inequality

$$\begin{aligned} \|\psi\|_r &\leq C_r \|\psi\|_2^{\frac{6-r}{2r}} (\|\partial_{x_1}\psi\|_2 + \|\psi\|_2)^{\frac{r-2}{2r}} (\|\partial_{x_2}\psi\|_2 + \|\psi\|_2)^{\frac{r-2}{2r}} (\|\partial_{x_3}\psi\|_2 + \|\psi\|_2)^{\frac{r-2}{2r}} \\ &\leq C_r \|\psi\|_2^{\frac{6-r}{2r}} \|\psi\|_{H^1(\Omega)}^{\frac{3(r-2)}{2r}}, \end{aligned} \quad (10)$$

for every $\psi \in H^1(\Omega)$ and every $r \in [2, 6]$. Here, Ω is the periodic box $[0, L]^3$. We remark that one can apply inequality (10) instead of inequality (9), and the methods presented in this paper to establish the same results in the case of periodic boundary conditions. The details of the proof in the periodic case are omitted.

2. THE MAIN RESULT

In this section we will prove our main result, which states that the strong solution to system (1)–(3) exists on the interval $[0, T]$ provided the assumption (7) or (8) holds.

Theorem 1. *Let $u_0 \in V$, and let $u = (u_1, u_2, u_3)$ be a Leray–Hopf weak solution to 3D NSE, system (1)–(3), with the initial value u_0 . Let $T > 0$, and suppose that, for some k, j , with $1 \leq k, j \leq 3$, u satisfies the condition (7) or (8), namely,*

$$\int_0^T \left\| \frac{\partial u_j(s)}{\partial x_k} \right\|_\alpha^\beta ds \leq M; \quad \text{when } k \neq j, \text{ and where } \alpha > 3, 1 \leq \beta < \infty, \text{ and } \frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{\alpha+3}{2\alpha}, \quad (11)$$

or

$$\int_0^T \left\| \frac{\partial u_j(s)}{\partial x_j} \right\|_\alpha^\beta ds \leq M; \quad \text{where } \alpha > 2, 1 \leq \beta < \infty, \text{ and } \frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{3(\alpha+2)}{4\alpha}, \quad (12)$$

for some $M > 0$. Then u is a strong solution of 3D NSE, system (1)–(3), on the interval $[0, T]$. Moreover, it is the only weak solution on $[0, T]$ with the initial datum u_0 .

Before we prove the main Theorem 1, we show the following two lemmas.

Lemma 2.

$$\left| \int_{\mathbb{R}^3} \phi f g dx_1 dx_2 dx_3 \right| \leq C \|\phi\|_2^{\frac{r-1}{r}} \|\partial_{x_1}\phi\|_{\frac{2}{3-r}}^{1/r} \|f\|_2^{\frac{r-2}{r}} \|\partial_{x_2}f\|_2^{1/r} \|\partial_{x_3}f\|_2^{1/r} \|g\|_2, \quad (13)$$

where $2 < r < 3$.

Proof. Observe, first, that it is enough to prove the inequality for functions $\phi, f, g \in C_0^\infty(\mathbb{R}^3)$ and then passing to the limit using a density argument.

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} \phi f g \, dx_1 dx_2 dx_3 \right| &\leq C \int_{\mathbb{R}^2} \left[\max_{x_1} |\phi| \left(\int_{\mathbb{R}} f^2 dx_1 \right)^{1/2} \left(\int_{\mathbb{R}} g^2 dx_1 \right)^{1/2} \right] dx_2 dx_3 \\
&\leq C \left[\int_{\mathbb{R}^2} \left(\max_{x_1} |\phi| \right)^r dx_2 dx_3 \right]^{1/r} \left[\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f^2 dx_1 \right)^{\frac{r}{r-2}} dx_2 dx_3 \right]^{\frac{r-2}{2r}} \left(\int_{\mathbb{R}^3} g^2 dx_1 dx_2 dx_3 \right)^{1/2} \\
&\leq C \left[\int_{\mathbb{R}^3} |\phi|^{r-1} |\partial_{x_1} \phi| \, dx_1 dx_2 dx_3 \right]^{1/r} \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} f^{\frac{2r}{r-2}} dx_2 dx_3 \right)^{\frac{r-2}{r}} dx_1 \right]^{\frac{1}{2}} \|g\|_2 \\
&\leq C \|\phi\|_2^{\frac{r-1}{2}} \|\partial_{x_1} \phi\|_{\frac{2}{3-r}}^{1/r} \|f\|_2^{\frac{r-2}{2}} \|\partial_{x_2} f\|_2^{1/r} \|\partial_{x_3} f\|_2^{1/r} \|g\|_2.
\end{aligned}$$

□

Lemma 3.

$$\left| \int_{\mathbb{R}^3} \phi f g \, dx_1 dx_2 dx_3 \right| \leq C \|\phi\|_2^{\frac{r-1}{2}} \|\partial_{x_3} \phi\|_{\frac{2}{3-r}}^{1/r} \|f\|_2^{\frac{r-2}{2}} \|\partial_{x_1} f\|_2^{1/r} \|\partial_{x_2} f\|_2^{1/r} \|g\|_2, \quad (14)$$

where $2 < r < 3$.

Proof. Here, again, it is enough to prove the inequality for functions $\phi, f, g \in C_0^\infty(\mathbb{R}^3)$.

$$\begin{aligned}
\left| \int_{\mathbb{R}^3} \phi f g \, dx_1 dx_2 dx_3 \right| &\leq C \int_{\mathbb{R}^2} \left[\max_{x_3} |\phi| \left(\int_{\mathbb{R}} f^2 dx_3 \right)^{1/2} \left(\int_{\mathbb{R}} g^2 dx_3 \right)^{1/2} \right] dx_1 dx_2 \\
&\leq C \left[\int_{\mathbb{R}^2} \left(\max_{x_3} |\phi| \right)^r dx_1 dx_2 \right]^{1/r} \left[\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}} f^2 dx_3 \right)^{\frac{r}{r-2}} dx_1 dx_2 \right]^{\frac{r-2}{2r}} \left(\int_{\mathbb{R}^3} g^2 dx_1 dx_2 dx_3 \right)^{1/2} \\
&\leq C \left[\int_{\mathbb{R}^3} |\phi|^{r-1} |\partial_{x_3} \phi| \, dx_1 dx_2 dx_3 \right]^{1/r} \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} f^{\frac{2r}{r-2}} dx_1 dx_2 \right)^{\frac{r-2}{r}} dx_3 \right]^{\frac{1}{2}} \|g\|_2 \\
&\leq C \|\phi\|_2^{\frac{r-1}{2}} \|\partial_{x_3} \phi\|_{\frac{2}{3-r}}^{1/r} \|f\|_2^{\frac{r-2}{2}} \|\partial_{x_1} f\|_2^{1/r} \|\partial_{x_2} f\|_2^{1/r} \|g\|_2.
\end{aligned}$$

□

Proof of the Theorem 1. Without loss of generality, we will assume that $j = 3$ and $k = 1$ in (11) and (12), namely,

$$\int_0^T \left\| \frac{\partial u_3(s)}{\partial x_1} \right\|_\alpha^\beta ds \leq M; \quad \text{where } \alpha > 3, 1 \leq \beta < \infty, \text{ and } \frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{\alpha+3}{2\alpha}, \quad (15)$$

or,

$$\int_0^T \left\| \frac{\partial u_3(s)}{\partial x_3} \right\|_\alpha^\beta ds \leq M; \quad \text{where } \alpha > 2, 1 \leq \beta < \infty, \text{ and } \frac{3}{\alpha} + \frac{2}{\beta} \leq \frac{3(\alpha+2)}{4\alpha}. \quad (16)$$

It is well-known that there exists a global in time Leray–Hopf weak solution to the 3D NSE, the system (1)–(3), in whole space \mathbb{R}^3 if $u_0 \in H$ (see, e.g., [7], [21], [23], [25], [32], [36]). It is also well-known that there exists a unique strong solution for a short time interval if $u_0 \in V$. In addition, this strong solution is the only weak solution, with the initial datum u_0 , on the maximal interval of existence of the strong solution.

Suppose that u is the strong solution with the initial value $u_0 \in V$ such that $u \in C([0, \mathcal{T}^*), V) \cap L^2([0, \mathcal{T}^*), H^2)$, where $[0, \mathcal{T}^*)$ is the maximal interval of existence of the unique strong solution. If $\mathcal{T}^* \geq T$ then there is nothing to prove. If, on the other hand, $\mathcal{T}^* < T$ our strategy is to show that the H^1 norm of this strong solution is bounded

uniformly in time over the interval $[0, \mathcal{T}^*)$, provided condition (15) or (16) is valid. As a result the interval $[0, \mathcal{T}^*)$ can not be a maximal interval of existence, and consequently $\mathcal{T}^* \geq T$. Which will conclude our proof.

From now on we focus on the strong solution, u , on its maximal interval of existence $[0, \mathcal{T}^*)$, where we assume that $\mathcal{T}^* < T$. As we have observed earlier the strong solution u will also be the only weak solution on the interval $[0, \mathcal{T}^*)$. Therefore, by the energy inequality (6), for Leray–Hopf weak solutions, we have (see, e.g., [7], [32], [35] or [36] for details)

$$\|u(t)\|_2^2 + \nu \int_0^t \|\nabla u(s)\|_2^2 ds \leq K_1, \quad (17)$$

for all $t \geq 0$, where

$$K_1 = \|u_0\|_2^2. \quad (18)$$

Next, let us show that the H^1 norm of the strong solution u is bounded on interval $[0, \mathcal{T}^*)$.

2.1. $\|\nabla_h u\|_2$ estimates. First we obtain some estimates of the horizontal gradient. Taking the inner product of the equation (1) with $-\Delta_h u$ in L^2 , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d\|\nabla_h u\|_2^2}{dt} + \nu \|\nabla_h \nabla u\|_2^2 \\ &= \int_{\mathbb{R}^3} [(u \cdot \nabla)u] \cdot \Delta_h u \, dx_1 dx_2 dx_3. \end{aligned}$$

By integration by parts few times, and using the incompressibility condition (2), we get

$$\begin{aligned} & - \int_{\mathbb{R}^3} (u \cdot \nabla)u \cdot \Delta_h u \, dx_1 dx_2 dx_3 = \int_{\mathbb{R}^3} \sum_{k,j=1}^3 \sum_{l=1}^2 \frac{\partial u_k}{\partial x_j} \frac{\partial u_j}{\partial x_l} \frac{\partial u_k}{\partial x_l} \, dx_1 dx_2 dx_3 \\ &= \int_{\mathbb{R}^3} \left\{ \left(\frac{\partial u_1}{\partial x_1} \right)^3 + \left(\frac{\partial u_2}{\partial x_2} \right)^3 + \left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) \left[\left(\frac{\partial u_1}{\partial x_2} \right)^2 + \left(\frac{\partial u_2}{\partial x_1} \right)^2 + \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \right] \right. \\ & \quad \left. + \sum_{k,l=1}^2 \frac{\partial u_k}{\partial x_3} \frac{\partial u_3}{\partial x_l} \frac{\partial u_k}{\partial x_l} + \sum_{j,l=1}^2 \frac{\partial u_3}{\partial x_j} \frac{\partial u_j}{\partial x_l} \frac{\partial u_3}{\partial x_l} + \sum_{l=1}^2 \frac{\partial u_3}{\partial x_3} \frac{\partial u_3}{\partial x_l} \frac{\partial u_3}{\partial x_l} \right\} dx_1 dx_2 dx_3 \\ &= \int_{\mathbb{R}^3} \left\{ - \frac{\partial u_3}{\partial x_3} \left[\left(\frac{\partial u_1}{\partial x_1} \right)^2 + \left(\frac{\partial u_2}{\partial x_2} \right)^2 - \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} + \left(\frac{\partial u_1}{\partial x_2} \right)^2 + \left(\frac{\partial u_2}{\partial x_1} \right)^2 + \frac{\partial u_1}{\partial x_2} \frac{\partial u_2}{\partial x_1} \right] \right. \\ & \quad \left. + \sum_{l=1}^2 \frac{\partial u_3}{\partial x_l} \left[\sum_{k=1}^2 \frac{\partial u_k}{\partial x_3} \frac{\partial u_k}{\partial x_l} + \sum_{k=1}^2 \frac{\partial u_3}{\partial x_k} \frac{\partial u_k}{\partial x_l} + \frac{\partial u_3}{\partial x_3} \frac{\partial u_3}{\partial x_l} \right] \right\} dx_1 dx_2 dx_3 \\ &\leq C \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla_h \nabla u| \, dx_1 dx_2 dx_3. \end{aligned}$$

Next, we will estimate the right-hand side of the above inequality using either Lemma 2 or Lemma 3. Each will be used for dealing with either one of the conditions (15) or (16). On the one hand by applying (13), with $\phi = |u_3|$, $f = |\nabla u|$, $g = |\nabla_h \nabla u|$, and $r = \frac{3\alpha-2}{\alpha}$, we get

$$\begin{aligned} & \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla_h \nabla u| \, dx_1 dx_2 dx_3 \\ &\leq C \|u_3\|_2^{\frac{2(\alpha-1)}{3\alpha-2}} \|\partial_{x_1} u_3\|_{\alpha}^{\frac{\alpha}{3\alpha-2}} \|\nabla u\|_2^{\frac{\alpha-2}{3\alpha-2}} \|\partial_{x_2} \nabla u\|_2^{\frac{\alpha}{3\alpha-2}} \|\partial_{x_3} \nabla u\|_2^{\frac{\alpha}{3\alpha-2}} \|\nabla \nabla_h u\|_2 \\ &\leq C \|u_3\|_2^2 \|\partial_{x_1} u_3\|_{\alpha}^{\frac{\alpha}{\alpha-1}} \|\nabla u\|_2^{\frac{\alpha-2}{\alpha-1}} \|\partial_{x_3} \nabla u\|_2^{\frac{\alpha}{\alpha-1}} + \frac{\nu}{2} \|\nabla \nabla_h u\|_2^2. \end{aligned} \quad (19)$$

On the other hand by applying (14), with $\phi = |u_3|$, $f = |\nabla u|$, $g = |\nabla_h \nabla u|$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla_h \nabla u| dx_1 dx_2 dx_3 \\ & \leq C \|u_3\|_2^{\frac{2(\alpha-1)}{3\alpha-2}} \|\partial_{x_3} u_3\|_{\alpha}^{\frac{\alpha}{3\alpha-2}} \|\nabla u\|_2^{\frac{\alpha-2}{3\alpha-2}} \|\partial_{x_1} \nabla u\|_2^{\frac{\alpha}{3\alpha-2}} \|\partial_{x_2} \nabla u\|_2^{\frac{\alpha}{3\alpha-2}} \|\nabla \nabla_h u\|_2 \\ & \leq C \|u_3\|_2^{\frac{4(\alpha-1)}{\alpha-2}} \|\partial_{x_3} u_3\|_{\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_2^2 + \frac{\nu}{2} \|\nabla \nabla_h u\|_2^2. \end{aligned} \quad (20)$$

In case we use (19) we obtain

$$\frac{d\|\nabla_h u\|_2^2}{dt} + \nu \|\nabla_h \nabla u\|_2^2 \leq C \|u_3\|_2^2 \|\partial_{x_1} u_3\|_{\alpha}^{\frac{\alpha-2}{\alpha-1}} \|\nabla u\|_2^{\frac{\alpha-2}{\alpha-1}} \|\partial_{x_3} \nabla u\|_2^{\frac{\alpha-2}{\alpha-1}}; \quad (21)$$

Alternatively, if we use (20), we obtain

$$\frac{d\|\nabla_h u\|_2^2}{dt} + \nu \|\nabla_h \nabla u\|_2^2 \leq C \|u_3\|_2^{\frac{4(\alpha-1)}{\alpha-2}} \|\partial_{x_3} u_3\|_{\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_2^2. \quad (22)$$

Therefore, integrating (21) and using Hölder inequality and applying (17) we get

$$\begin{aligned} & \|\nabla_h u(t)\|_2^2 + \nu \int_0^t \|\nabla_h \nabla u(s)\|_2^2 ds \\ & \leq \|\nabla_h u_0\|_2^2 + C \left(\int_0^t \|\partial_{x_1} u_3(s)\|_{\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u(s)\|_2^2 ds \right)^{\frac{\alpha-2}{2(\alpha-1)}} \left(\int_0^t \|\Delta u(s)\|_2^2 ds \right)^{\frac{\alpha}{2(\alpha-1)}}, \end{aligned} \quad (23)$$

for all $t \in [0, \mathcal{T}^*)$. Alternatively, integrating (22) and using Hölder inequality and applying (17) we get a different estimate

$$\|\nabla_h u(t)\|_2^2 + \nu \int_0^t \|\nabla_h \nabla u(s)\|_2^2 ds \leq \|\nabla_h u_0\|_2^2 + C \int_0^t \left(\|\partial_{x_3} u_3(s)\|_{\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u(s)\|_2^2 \right) ds \quad (24)$$

for all $t \in [0, \mathcal{T}^*)$.

2.2. $\|\nabla u\|_2$ estimates. Taking the inner product of the equation (1) with $-\Delta u$ in L^2 , we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d\|\nabla u\|_2^2}{dt} + \nu \|\Delta u\|_2^2 \\ & = \int_{\mathbb{R}^3} [(u \cdot \nabla)u] \cdot \Delta_h u dx_1 dx_2 dx_3 + \int_{\mathbb{R}^3} [(u \cdot \nabla)u] \cdot \partial_{x_3}^2 u dx_1 dx_2 dx_3 \\ & \leq C \int_{\mathbb{R}^3} \left[|u_3| |\nabla u| |\nabla_h \nabla u| + |\nabla_h u| |\partial_{x_3} u|^2 \right] dx_1 dx_2 dx_3. \end{aligned} \quad (25)$$

Applying the Cauchy–Schwarz inequality and (9), with $r = 4$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^3} |\nabla_h u| |\partial_{x_3} u|^2 dx_1 dx_2 dx_3 \leq \|\nabla_h u\|_2 \|\nabla u\|_4^2 \\ & \leq \|\nabla_h u\|_2 \|\nabla u\|_2^{1/2} \|\nabla_h \nabla u\|_2 \|\Delta u\|_2^{1/2}. \end{aligned} \quad (26)$$

Now, we are ready to complete our proof for the case when condition (15) holds. By (19) and Young's inequality, we have

$$\begin{aligned} & \int_{\mathbb{R}^3} |u_3| |\nabla u| |\nabla_h \nabla u| dx_1 dx_2 dx_3 \\ & \leq C \|u_3\|_2^2 \|\partial_{x_1} u_3\|_{\alpha}^{\frac{\alpha-1}{\alpha-2}} \|\nabla u\|_2^{\frac{\alpha-2}{\alpha-1}} \|\partial_{x_3} \nabla u\|_2^{\frac{\alpha}{\alpha-1}} + \frac{\nu}{2} \|\nabla \nabla_h u\|_2^2 \\ & \leq C \|u_3\|_2^{\frac{4(\alpha-1)}{\alpha-2}} \|\partial_{x_1} u_3\|_{\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u\|_2^2 + \frac{3\nu}{4} \|\Delta u\|_2^2. \end{aligned}$$

As a result of the above and (25), we get

$$\begin{aligned} & \frac{d\|\nabla u\|_2^2}{dt} + \frac{\nu}{2}\|\Delta u\|_2^2 \\ & \leq C\|\nabla_h u\|_2\|\nabla u\|_2^{1/2}\|\nabla_h \nabla u\|_2\|\Delta u\|_2^{1/2} + C\|u_3\|_2^{\frac{4(\alpha-1)}{\alpha-2}}\|\partial_{x_1} u_3\|_{\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u\|_2^2; \end{aligned}$$

Integrating the above inequality and using Hölder inequality we obtain

$$\begin{aligned} & \|\nabla u(t)\|_2^2 + \frac{\nu}{2}\int_0^t\|\Delta u\|_2^2 ds \\ & \leq \|\nabla u(0)\|_2^2 + C\left(\sup_{0\leq s\leq t}\|\nabla_h u(s)\|_2\right)\left(\int_0^t\|\nabla u\|_2^2 ds\right)^{\frac{1}{4}}\left(\int_0^t\|\nabla_h \nabla u(s)\|_2^2 ds\right)^{\frac{1}{2}}\left(\int_0^t\|\Delta u(s)\|_2^2 ds\right)^{\frac{1}{4}} \\ & \quad + C\left(\sup_{0\leq s\leq t}\|u(s)\|_2^{\frac{4(\alpha-1)}{\alpha-2}}\right)\left(\int_0^t\|\partial_{x_3} u_3(s)\|_{\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u(s)\|_2^2 ds\right). \end{aligned}$$

Thanks to (17) and (23), we get

$$\begin{aligned} & \|\nabla u(t)\|_2^2 + \frac{\nu}{2}\int_0^t\|\Delta u\|_2^2 ds \\ & \leq \|\nabla u(0)\|_2^2 + CK_1^{1/4}\left[\|\nabla_h u_0\|_2^2 + C\left(\int_0^t\|\partial_{x_1} u_3(s)\|_{\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u(s)\|_2^2 ds\right)^{\frac{\alpha-2}{2(\alpha-1)}}\left(\int_0^t\|\Delta u(s)\|_2^2 ds\right)^{\frac{\alpha}{2(\alpha-1)}+\frac{1}{4}}\right] \\ & \quad + CK_1^{\frac{2(\alpha-1)}{\alpha-2}}\left(\int_0^t\|\partial_{x_1} u_3(s)\|_{\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u(s)\|_2^2 ds\right). \end{aligned}$$

By Young's and Hölder inequalities, we obtain

$$\begin{aligned} & \|\nabla u(t)\|_2^2 + \frac{\nu}{4}\int_0^t\|\Delta u\|_2^2 ds \\ & \leq C\|\nabla u(0)\|_2^2 + C\left(\int_0^t\|\partial_{x_1} u_3(s)\|_{\alpha}^{\frac{4\alpha}{\alpha-3}}\|\nabla u(s)\|_2^2 ds\right)\left(\int_0^t\|\nabla u(s)\|_2^2 ds\right)^{\frac{\alpha-1}{2(\alpha-2)}} \\ & \quad + CK_1^{\frac{2(\alpha-1)}{\alpha-2}}\left(\int_0^t\|\partial_{x_1} u_3(s)\|_{\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u(s)\|_2^2 ds\right). \end{aligned} \tag{27}$$

Thanks again to (17), we get

$$\|\nabla u(t)\|_2^2 + \frac{\nu}{4}\int_0^t\|\Delta u\|_2^2 ds \leq C\|\nabla u(0)\|_2^2 + C\int_0^t\|\partial_{x_1} u_3(s)\|_{\alpha}^{\frac{4\alpha}{\alpha-3}}\|\nabla u(s)\|_2^2 ds. \tag{28}$$

Therefore, in case (11) holds, we apply Gronwall inequality to obtain

$$\|\nabla u(t)\|_2^2 + \nu\int_0^t\|\Delta u\|_2^2 ds \leq C(1 + \|\nabla u(0)\|_2^2)e^{CM},$$

for all $t \in [0, \mathcal{T}^*)$. Therefore, if the condition (15) holds the H^1 norm of the solution u is bounded, and this completes our proof in this case. Next, we complete the proof when u_3 satisfies (16). Thanks to (25), (20) and (26), we get

$$\begin{aligned} & \frac{d\|\nabla u\|_2^2}{dt} + \frac{\nu}{2}\|\Delta u\|_2^2 \\ & \leq C\|\nabla_h u\|_2\|\nabla u\|_2^{1/2}\|\nabla_h \nabla u\|_2\|\Delta u\|_2^{1/2} + C\|u_3\|_2^{\frac{4(\alpha-1)}{\alpha-2}}\|\partial_{x_3} u_3\|_{\alpha}^{\frac{2\alpha}{\alpha-2}}\|\nabla u\|_2^2; \end{aligned}$$

Integrating the above inequality and using Hölder inequality we obtain

$$\begin{aligned} & \|\nabla u(t)\|_2^2 + \frac{\nu}{2} \int_0^t \|\Delta u\|_2^2 ds \\ & \leq \|\nabla u(0)\|_2^2 + C \left(\sup_{0 \leq s \leq t} \|\nabla_h u(s)\|_2 \right) \left(\int_0^t \|\nabla u\|_2^2 ds \right)^{\frac{1}{4}} \left(\int_0^t \|\nabla_h \nabla u(s)\|_2^2 ds \right)^{\frac{1}{2}} \left(\int_0^t \|\Delta u(s)\|_2^2 ds \right)^{\frac{1}{4}} \\ & + C \left(\sup_{0 \leq s \leq t} \|u(s)\|_2^{\frac{4(\alpha-1)}{\alpha-2}} \right) \left(\int_0^t \|\partial_{x_3} u_3(s)\|_{\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u(s)\|_2^2 ds \right). \end{aligned}$$

Thanks to (17) and (24), we get

$$\begin{aligned} & \|\nabla u(t)\|_2^2 + \frac{\nu}{2} \int_0^t \|\Delta u\|_2^2 ds \\ & \leq \|\nabla u(0)\|_2^2 + CK_1^{1/4} \left[\|\nabla_h u_0\|_2^2 + C \left(\int_0^t \|\partial_{x_3} u_3(s)\|_{\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u(s)\|_2^2 ds \right) \left(\int_0^t \|\Delta u(s)\|_2^2 ds \right)^{\frac{1}{4}} \right] \\ & + CK_1^{\frac{2(\alpha-1)}{\alpha-2}} \left(\int_0^t \|\partial_{x_1} u_3(s)\|_{\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u(s)\|_2^2 ds \right). \end{aligned}$$

By Young's and Hölder inequalities, we obtain

$$\begin{aligned} & \|\nabla u(t)\|_2^2 + \frac{\nu}{4} \int_0^t \|\Delta u\|_2^2 ds \\ & \leq C \|\nabla u(0)\|_2^2 + C \left(\int_0^t \|\partial_{x_3} u_3(s)\|_{\alpha}^{\frac{8\alpha}{3(\alpha-2)}} \|\nabla u(s)\|_2^2 ds \right) \left(\int_0^t \|\nabla u(s)\|_2^2 ds \right)^{\frac{1}{4}} \\ & + CK_1^{\frac{2(\alpha-1)}{\alpha-2}} \left(\int_0^t \|\partial_{x_3} u_3(s)\|_{\alpha}^{\frac{2\alpha}{\alpha-2}} \|\nabla u(s)\|_2^2 ds \right). \end{aligned} \quad (29)$$

Thanks again to (17), we get

$$\|\nabla u(t)\|_2^2 + \frac{\nu}{4} \int_0^t \|\Delta u\|_2^2 ds \leq C \|\nabla u(0)\|_2^2 + C \left(\int_0^t \|\partial_{x_3} u_3(s)\|_{\alpha}^{\frac{8\alpha}{3(\alpha-2)}} \|\nabla u(s)\|_2^2 ds \right). \quad (30)$$

Therefore, by Gronwall inequality and (16) we obtain

$$\|\nabla u(t)\|_2^2 + \nu \int_0^t \|\Delta u\|_2^2 ds \leq C(1 + \|\nabla u(0)\|_2^2) e^{CM}.$$

for all $t \in [0, \mathcal{T}^*)$. Therefore, the H^1 norm of the strong solution u is bounded on the maximal interval of existence $[0, \mathcal{T}^*)$. This completes the proof of Theorem 1.

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