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Gevrey Regularity for Nonlinear Analytic Parabolic Equations on the Sphere

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The regularity of solutions to a large class of analytic nonlinear parabolic equations on the two-dimensional sphere is considered. In particular, it is shown that these solutions belong to a certain Gevrey class of functions, which is a subset of the set of real analytic functions. As a consequence it can be shown that the Galerkin schemes, based on the spherical harmonics, converge exponentially fast to the exact solutions, as the number of modes involved in the approximation tends to infinity. Furthermore, in the case that the underlying evolution equation has a global attractor, then this global attractor is contained in the space of spatially real analytic functions whose radii of analyticity are bounded uniformly from below.

KEY WORDS: Gevrey regularity; nonlinear analytic parabolic equations; sphere.

1. INTRODUCTION

Parabolic evolution partial differential equations are known to exhibit an instantaneous smoothing effect; that is, for general initial data their solutions become instantaneously smoother than the initial data. Foias and Temam [7] developed an energy technique, which involves certain pseudo-differential operators, to prove that the solutions of the Navier–Stokes equations, starting with initial data in the Sobolev space H^1, become

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instantaneously elements of a certain Gevrey class of regularity, which, in particular, makes them real analytic functions. There are two key points in the proof of Foias and Temam [7] that one has to keep in mind while generalizing their approach. First, the authors take advantage of the multiplicative (in this case quadratic) nature of the nonlinearity in the Navier–Stokes equations. Second, the absence of physical boundaries: in [7] they consider the case of periodic boundary conditions. Indeed, Promislow [21] observed that one can generalize this approach to nonlinear parabolic evolution equations with polynomial nonlinearities, subject to periodic boundary conditions. Later, this approach was generalized further in [6] to include nonlinear parabolic evolution equations with general analytic nonlinearities, again subject to periodic boundary conditions. In this paper, we take these two key points into consideration and prove the instantaneous Gevrey regularity result for parabolic nonlinear analytic equations on the two-dimensional sphere.

This instantaneous forward smoothing effect that parabolic partial differential equations enjoy is in a sense equivalent to the backward ill-posedness from which they suffer. However, in the case that the underlying system has a global attractor (see, e.g., [8, 25]), then the solutions exist backward in time for initial values in the global attractor. As a result it follows easily that the global attractor is contained in a certain Gevrey class of regularity, and in particular, it is contained in the space of spatially real analytic functions with a uniform lower bound on their radii of analyticity. Let us remark that the estimates provided by this approach for the uniform lower bound for the radii of analyticity is sharp (see [20]). Moreover, it is worth mentioning that there are examples of dissipative PDEs which are not parabolic, and in particular, they do not enjoy this instantaneous smoothing effect, for which their global attractors are contained in a certain Gevrey class of regularity (see [18, 19]). In a sense, for such equations there is smoothing, however, it occurs only as time goes to infinity. The proof of the Gevrey regularity of the global attractor for such equations is not as straightforward as in the case of parabolic PDEs (see [18, 19]).

In this paper we consider the initial value problem on the two-dimensional sphere $S \equiv S^2$:

$$u_t - \nu \Delta u + G(u, \nabla u) = 0 \tag{1.1}$$

$$u(x, 0) = u^0 \tag{1.2}$$

where ν is a positive constant, G is an analytic function, and Δ is the Laplace operator on the sphere.

The analogue of (1.1), (1.2) with periodic boundary conditions has been studied in [6]. Specifically, it has been shown in [6] that, if the initial data belong to the Sobolev space H^p_{per}, with $p > n/2$, then the initial boundary value problem associated with (1.1) and (1.2) in $[0, l]^n$ with periodic boundary conditions has a unique solution that lies in the Gevrey class of functions $\mathcal{D}(A^{p/2}e^{tA^{1/2}})$, where $A = -\Delta + Id$. One of the main technical points in [6] was showing that the Gevrey class $G^{p/2}_\sigma = \mathcal{D}(A^{p/2}e^{\sigma A^{1/2}})$ is a topological multiplicative algebra, whenever $\sigma \geqslant 0$ and $p > n/2$. The key point in their proof (cf. Lemma 1, Ref. 6) lies in the simple fact that $e^{ik \cdot x}$, for $k \in \mathbb{Z}^n$, is an eigenfunction for the Laplace operator and that the product $e^{ik \cdot x}e^{im \cdot x} = e^{i(k+m) \cdot x}$ is also an eigenfunction, for every $k, m \in \mathbb{Z}^n$.

Inspired by the work of Foias and Temam [7] and its generalizations [6] and [21], we prove in this article a regularity result for the solution of (1.1) and (1.2) under some restrictions on the nonlinearity G. More precisely, we prove that if the initial data belong to the Sobolev space $H^p(S)$, for $p > 3/2$, then (1.1) and (1.2) have a unique solution that lies in the Gevrey class $\mathcal{D}(A^{p/2}e^{tA^{1/2}})$ for $t \in [0, T]$, where $A = -\Delta + Id$ and $T > 0$, is the local existence time, depending only on ν and the $H^p(S)$ norm of u^0.

The main technical difficulty in this paper is proving that the Gevrey class $G^{p/2}_\sigma = \mathcal{D}(A^{p/2}e^{\sigma A^{1/2}})$ is a topological multiplicative algebra, whenever $\sigma \geqslant 0$ and $p > 3/2$. This difficulty lies in the fact that the product of two spherical harmonics (the eigenfunctions for Δ on the two-dimensional sphere) is a linear combination of spherical harmonics and not just a single eigenfunction, as in the periodic boundary conditions case. In fact, this property of the spherical harmonics was the main contributor to the unexpected restriction in Lemma 1 (Section 3), that $p > 3/2$.

The plan of this paper is as follows. In Section 2, we present technical definitions, notation, and well – known properties for spherical harmonics. Section 3 is the main section, and it is devoted to the existence and regularity of solutions.

2. PRELIMINARIES

The main purpose of this section is to introduce the technical definitions and notations. Naturally, we consider the spherical coordinate system $(x^1, x^2) \equiv (\theta, \phi)$ on S, where $\theta \in (0, \pi)$ is the latitude and $\phi \in (0, 2\pi)$ is the longitude. Let $\vec{n} = (\sin x^1 \cos x^2, \sin x^1 \sin x^2, \cos x^1)$ be the outward normal vector to S in $\mathbb{R}^3$. The vectors $\vec{n}^1 = \partial\vec{n}/\partial x^1$ and $\vec{n}^2 = \partial\vec{n}/\partial x^2$ form a basis for **TS**, the tangent space to S. For each $\xi \in \mathbf{TS}$ we write $\xi = (\xi^i)$, where $\xi = \xi^1\vec{n}^1 + \xi^2\vec{n}^2$. The Riemannian metric of the sphere with respect to this parameterization is given by

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 x^1 \end{pmatrix}$$

The inverse of (g_{ij}) is denoted (g^{ij}), i.e., $g_{ij}g^{jk}=\delta_j^k$ (here and later we use the Einstein's convention for summation over successive indices). The inner product on **TS** is given by

$$\xi\cdot\eta=g_{ij}\xi^i\eta^j, \qquad |\xi|=(\xi\cdot\xi)^{1/2}, \qquad \text{for} \quad \xi,\eta\in\mathbf{TS}$$

To every vector $\xi=(\xi^i)\in\mathbf{TS}$ we assign the covector $\tilde{\xi}=(\tilde{\xi}_i)$, where $\tilde{\xi}_i=g_{ij}\xi^j$, and to each covector $\eta=(\eta_i)$ we assign the vector $\hat{\eta}=(\hat{\eta}^j)$, where $\hat{\eta}^j=g^{ji}\eta_i$.

If $f\in C^\infty(S)$, then we define its gradient ∇f in $C^\infty(\mathbf{TS})$ by $\nabla f=((\nabla f)^i)=(g^{ij}(\partial f/\partial x^j))$, i.e., $\nabla f=(\partial f/\partial x^1)\,\vec{n}^1+(1/\sin^2 x^1)(\partial f/\partial x^2)\,\vec{n}^2$. For every $u=(u^i)\in C^\infty(\mathbf{TS})$ we define its covariant derivative by

$$\nabla_j u^i=\frac{\partial u^i}{\partial x^j}+\Gamma^i_{kj}u^k$$

where Γ^i_{kj} is the Christoffel symbol given by

$$\Gamma^i_{kj}=\frac{1}{2}\,g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial x^k}+\frac{\partial g_{k\ell}}{\partial x^j}-\frac{\partial g_{kj}}{\partial x^\ell}\right)$$

The divergence of $u=(u^i)\in C^\infty(\mathbf{TS})$ is given by

$$\operatorname{div} u=\nabla_i u^i=\frac{1}{\sin x^1}\left[\frac{\partial}{\partial x^1}((\sin x^1)\,u^1)+\frac{\partial}{\partial x^2}((\sin x^1)\,u^2)\right]$$

The Laplacian of a scalar function $f\in C^\infty(S)$ is defined by

$$\Delta f=\operatorname{div}\nabla f=\frac{1}{\sin x^1}\frac{\partial}{\partial x^1}\left(\sin x^1\frac{\partial f}{\partial x^1}\right)+\frac{1}{\sin^2 x^1}\frac{\partial^2 f}{(\partial x^2)^2}$$

Let the space $L^p(S)$ be the standard L^p—Lebesgue space on S and the space $H^p(S)$ be the standard Sobolev space on S (cf. Adams [1] and Aubin [2]). For $L^2(S)$, we define the inner product by

$$\langle u,v\rangle_{L^2}=\int_S u\bar{v}\,dS \qquad \text{for} \quad u,v\in L^2(S)$$

here $dS=\sin\theta\,d\theta\,d\phi$ (the Lebesgue measure on S). Let $A=-\Delta+Id=-\operatorname{div}\nabla+Id\colon H^2(S)\to L^2(S)$, and $F(u,\nabla u)=G(u,\nabla u)-\nu u$, then (1.1) and (1.2) can be written

$$u_t+\nu Au+F(u,\nabla u)=0 \tag{2.1}$$

$$u(x,0)=u^0 \tag{2.2}$$

The following are well-known results concerning the operator A (cf. [10, 13, 15, 16, 18]).

(i) For each $n=0, 1, 2,...,$ the functions

$$Y_n^m = C_n^m e^{im\phi} P_n^m(\cos\theta), \qquad m=0, \pm 1, \pm 2,..., \pm n$$

are the eigenfunctions of A corresponding to the eigenvalue $\lambda_n = n^2+n+1$. The numbers C_n^m are the normalizers in $L^2(S)$, given by $C_n^m = [((2n+1)/4\pi)((n-|m|)!/(n+|m|)!]^{1/2}$, and P_n^m are the associated Legendre functions given by

$$P_n^m(x) = (1-x^2)^{m/2} \left(\frac{\partial}{\partial x}\right)^m P_n(x), \qquad m=0, 1, 2,..., \qquad \text{and} \qquad |x| \leqslant 1$$

where $P_n(\cdot)$ denotes the Legendre polynomial of degree n. Moreover, we define $P_n^{-m}(x) = P_n^m(x)$, for $m=0, 1, 2,..., n$.

(ii) The set $\{Y_n^m : m=0, \pm 1, \pm 2,..., \pm n;$ and $n=0, 1, 2,...\}$ forms an orthonormal basis for $L^2(S)$. Namely, for every $u \in L^2(S)$, we have $u = \sum_{n \geqslant 0} \sum_{|m| \leqslant n} u_n^m Y_n^m$, where the convergence is in $L^2(S)$, with $u_n^m = \int_S u \bar{Y}_n^m \, dS$ such that $u_n^m = \bar{u}_n^{-m}$, and $\|u\|_{L^2(S)}^2 = \sum_{n \geqslant 0} \sum_{|m| \leqslant n} |u_n^m|^2$.

Throughout the paper we use the following notation. $\sum_{m,n} = \sum_{n \geqslant 0} \sum_{|m| \leqslant n}$, $L^2 = L^2(S)$, $H^p = H^p(S)$, etc. The Sobolev spaces H^p are easily characterized by the domains of $A^{p/2}$, i.e.,

$$H^p \equiv \mathscr{D}(A^{p/2}) \equiv \left\{ u \in L^2 : u = \sum_{m,n} u_n^m Y_n^m, \ \sum_{m,n} \lambda_n^p \, |u_n^m|^2 < \infty \right\}$$

and the norm $\|u\|_{H^p}$ is equivalent to $(\sum_{m,n} \lambda_n^p \, |u_n^m|^2)^{1/2}$. Therefore, we set

$$\|u\|_{H^p}^2 \equiv \sum_{m,n} \lambda_n^p \, |u_n^m|^2$$

Finally, following [7], we introduce the special Gevrey classes of functions $G_\sigma^{p/2} = \mathscr{D}(A^{p/2} e^{\sigma A^{1/2}})$, for $\sigma > 0$. We remark that $G_\sigma^{p/2}$ is a Hilbert space with respect to the inner product

$$\langle u, v \rangle_{G_\sigma^{p/2}} = \sum_{m,n} \lambda_n^p e^{2\sigma \lambda_n^{1/2}} u_n^m \bar{v}_n^m \qquad \text{for} \quad u, v \in G_\sigma^{p/2}$$

and where $u = \sum_{m,n} u_n^m Y_n^m$, and $v = \sum_{m,n} v_n^m Y_n^m$. Also observe that the space of real analytic functions $C^\omega(S) = \bigcup_{\sigma > 0} G_\sigma^{p/2}$ (cf. John [9], p. 73).

3. EXISTENCE AND REGULARITY

3.1. Quasi-linear Nonlinearity

In this section we establish the existence of local regular solutions (see Definition 1 below) of Eqs. (2.1) and (2.2) under the assumption that the initial data $u_0 \in H^p$ for $\frac{3}{2} < p \leqslant \frac{5}{2}$. For reasons that will become clear in the proof (see also [6] for more details) we will require that the growth order of the function $F(u, \nabla u)$ with respect to the variable ∇u be less than two, for large ∇u. Since $F(u, \nabla u)$ is analytic, this restricts the function $F(u, \nabla u)$ to be at most linear in ∇u. Therefore, in this section we assume that

$$\begin{aligned}&\text{the nonlinearity } F(u, \nabla u) = (\vec{f}(u) \cdot \nabla u)^a + f_4(u),\\ &\text{where } \vec{f} = (f_1, f_2, f_3) \in \mathbb{R}^3 \text{ and } f_j,\ j = 1, 2, 3, 4 \text{ are given}\\ &\text{analytic functions, and } a = 0 \text{ or } 1\end{aligned} \tag{3.1}$$

$$\begin{aligned}&f_i(s) \text{ has a majorant } g_i(s),\ i = 1, 2, 3, 4, \text{ i.e., if } f_i(s) = \sum_{k=0}^{\infty} a_{ik} s^k\\ &\text{then } g_i(s) = \sum_{k=0}^{\infty} |a_{ik}|\, s^k \text{ converges for all } s \in \mathbb{R}\end{aligned} \tag{3.2}$$

Definition 1. Let $u^0 \in H^p$ with $p > 1$. A *regular solution* of (2.1) and (2.2) on $[0, T]$ is a function u such that

$$u \in C([0, T]; H^p) \cap L^2([0, T]; D(A)) \tag{3.3}$$

$$\frac{du}{dt} \in L^2([0, T]; L^2) \tag{3.4}$$

and

$$\begin{aligned}&\langle u(t), \phi \rangle_{L^2} - \langle u(t_0), \phi \rangle_{L^2} + \nu \int_{t_0}^{t} \langle A^{1/2} u(s), A^{1/2} \phi \rangle_{L^2}\, ds\\ &\quad + \int_{t_0}^{t} \langle F(u(s), \nabla u(s)), \phi \rangle_{L^2}\, ds = 0\end{aligned} \tag{3.5}$$

for every $\phi \in H^1$ and almost every $t,\ t_0 \in [0, T]$.

Our first main theorem is as follows.

Theorem 1. *Let $M_0 > 0$ be given and let $u^0 \in H^p$ with $p > 3/2$ such that $\|u^0\|_{H^p} \leqslant M_0$. Then, there exists a constant $T^* > 0$ depending only on M_0 and ν, such that under the assumptions (3.1) and (3.2) the problems (2.1)* *and (2.2) have a unique regular solution u on* $[0, T^*]$. *Moreover,* $u(t) \in G_t^{p/2}$ *for all* $t \in [0, T^*]$.

In order to prove the theorem, we need the following lemmas, which are similar to the ones in [6].

Lemma 1. *For* $\sigma \geqslant 0$, *and* $p > 3/2$, *the Hilbert space* $G_\sigma^{p/2}$ *is a topological algebra. Specifically, if u and* $v \in G_\sigma^{p/2}$, *then* $uv \in G_\sigma^{p/2}$, *and*

$$\|uv\|_{G_\sigma^{p/2}} \leqslant c_p \|u\|_{G_\sigma^{p/2}} \|v\|_{G_\sigma^{p/2}} \tag{3.6}$$

where c_p *is a positive constant depending only on p.*

Proof. Let $u, v \in G_\sigma^{p/2} = \mathscr{D}(A^{p/2} e^{\sigma A^{1/2}})$ with

$$u = \sum_{m=0}^{\infty} \sum_{j=-m}^{j=m} u_m^j Y_m^j, \qquad \text{and} \qquad v = \sum_{n=0}^{\infty} \sum_{q=-n}^{q=n} v_n^q Y_n^q$$

Let

$$u_m = \sum_{j=-m}^{j=m} u_m^j Y_m^j, \qquad \text{and} \qquad v_n = \sum_{q=-n}^{q=n} v_n^q Y_n^q$$

then we have $u = \sum_{m \geqslant 0} u_m$ and $v = \sum_{n \geqslant 0} v_n$. Let us set $Y_m^j = 0$, when $|j| > m$. Recall that (cf. [10], p. 16; [13], pp. 235–243)

$$Y_m^j Y_n^q = \sum_{k=0}^{\min\{m,n\}} A(m, n, j, q, k)\, Y_{m+n-2k}^{j+q} \tag{3.7}$$

and

$$\|Y_m^j Y_n^q\|_{L^2}^2 = \sum_{k=0}^{\min\{m,n\}} |A(m, n, j, q, k)|^2$$

where $A(m, n, j, q, k) = \int_S Y_m^j Y_n^q \bar{Y}_{m+n-2k}^{j+q}\, dS$. As a result, we have

$$\begin{aligned} u_m v_n &= \sum_{|j| \leqslant m} \sum_{|q| \leqslant n} u_m^j v_n^q Y_m^j Y_n^q \\ &= \sum_{|j| \leqslant m} \sum_{|q| \leqslant n} \sum_{k=0}^{\min\{m,n\}} u_m^j v_n^q A(m, n, j, q, k) Y_{m+n-2k}^{j+q} \end{aligned}$$

Denote by

$$w_{m+n-2k}^{m,n} = \sum_{|j| \leqslant m} \sum_{|q| \leqslant n} u_m^j v_n^q A(m, n, j, q, k) Y_{m+n-2k}^{j+q}$$

and note that

$$\int_S w^{m,n}_{m+n-2k} w^{m,n}_{m+n-2l}\, dS = 0 \qquad \text{when} \quad k \neq l$$

and that

$$A w^{m,n}_{m+n-2k} = \lambda_{m+n-2k} w^{m,n}_{m+n-2k}$$

Then we have

$$u_m v_n = \sum_{k=0}^{\min\{m,n\}} w^{m,n}_{m+n-2k}$$

and

$$\|u_m v_n\|^2_{L^2} = \sum_{k=0}^{\min\{m,n\}} \|w^{m,n}_{m+n-2k}\|^2_{L^2} \tag{3.8}$$

Thus we have

$$\begin{aligned}
uv &= \sum_{m\geqslant 0} \sum_{n\geqslant 0} u_m v_n = \sum_{m\geqslant 0} \sum_{n\geqslant 0} \sum_{k=0}^{\min\{m,n\}} w^{m,n}_{m+n-2k} \\
&= \sum_{s\geqslant 0} \sum_{m+n=2s} \sum_{k=0}^{\min\{m,n\}} w^{m,n}_{2s-2k} + \sum_{s\geqslant 0} \sum_{m+n=2s+1} \sum_{k=0}^{\min\{m,n\}} w^{m,n}_{2s+1-2k} \\
&= \sum_{s\geqslant 0} \left(\sum_{m=0}^{s} \sum_{k=0}^{m} w^{m,2s-m}_{2s-2k} + \sum_{n=0}^{s} \sum_{k=0}^{n} w^{2s-n,n}_{2s-2k} - \sum_{k=0}^{s} w^{s,s}_{2s-2k} \right) \\
&\quad + \sum_{s\geqslant 0} \left(\sum_{m=0}^{s} \sum_{k=0}^{m} w^{m,2s+1-m}_{2s+1-2k} + \sum_{n=0}^{s} \sum_{k=0}^{n} w^{2s+1-n,n}_{2s+1-2k} \right) \\
&= I_1 + I_2 - I_3 + I_4 + I_5
\end{aligned}$$

Now consider I_1. By changing the order of summations between indices m and k, we obtain

$$I_1 = \sum_{s\geqslant 0} \sum_{k=0}^{s} \sum_{m=k}^{s} w^{m,2s-m}_{2s-2k}$$

then by using the fact that $\sum_{s'\geqslant 0} \sum_{k\geqslant 0} =^{s=s'+k} \sum_{s\geqslant 0} \sum_{k=0}^{s}$, one has

$$I_1 = \sum_{s'\geqslant 0} \sum_{k\geqslant 0} \sum_{m=k}^{s'+k} w^{m,2s'+2k-m}_{2s'}$$

Finally, let $m' = m - k$, and replace s' and m' back to s and m, respectively; we have

$$I_1 = \sum_{s \geqslant 0} \sum_{k \geqslant 0} \sum_{m=0}^{s} w_{2s}^{m+k,\, 2s+k-m}$$

Similarly, one can show that

$$I_2 = \sum_{s \geqslant 0} \sum_{k \geqslant 0} \sum_{n=0}^{s} w_{2s}^{2s+k-n,\, n+k}$$

$$I_4 = \sum_{s \geqslant 0} \sum_{k \geqslant 0} \sum_{m=0}^{s} w_{2s+1}^{m+k,\, 2s+k+1-m}$$

$$I_5 = \sum_{s \geqslant 0} \sum_{k \geqslant 0} \sum_{n=0}^{s} w_{2s+1}^{2s+k+1-n,\, n+k}$$

Now I_3 can be treated similarly. By using the fact that $\sum_{s' \geqslant 0} \sum_{k \geqslant 0} =^{s=s'+k} \sum_{s \geqslant 0} \sum_{k=0}^{s}$, we obtain

$$I_3 = \sum_{s' \geqslant 0} \sum_{k \geqslant 0} w_{2s'}^{s'+k,\, s'+k} = \sum_{s \geqslant 0} \sum_{k \geqslant 0} w_{2s}^{s+k,\, s+k}$$

Thus, we have

$$uv = \sum_{s \geqslant 0} \sum_{k \geqslant 0} \sum_{m=0}^{s} \left(w_{2s+1}^{m+k,\, 2s+k+1-m} + w_{2s+1}^{2s+k+1-m,\, m+k}\right) + \sum_{s \geqslant 0} \sum_{k \geqslant 0} \left\{ \sum_{m=0}^{s} \left(w_{2s}^{m+k,\, 2s+k-m} + w_{2s}^{2s+k-m,\, m+k}\right) - w_{2s}^{s+k,\, s+k} \right\}$$

Since $A w_{m+n-2k}^{m,n} = \lambda_{m+n-2k} w_{m+n-2k}^{m,n}$, it follows that

$$\begin{aligned} &A^{p/2} e^{\sigma A^{1/2}}(uv) \\ &= \sum_{s \geqslant 0} \sum_{k \geqslant 0} \sum_{m=0}^{s} \lambda_{2s+1}^{p/2} e^{\sigma \lambda_{2s+1}^{1/2}} \left(w_{2s+1}^{m+k,\, 2s+k+1-m} + w_{2s+1}^{2s+k+1-m,\, m+k}\right) \\ &\quad + \sum_{s \geqslant 0} \sum_{k \geqslant 0} \lambda_{2s}^{p/2} e^{\sigma \lambda_{2s}^{1/2}} \\ &\quad \times \left\{ \sum_{m=0}^{s} \left(w_{2s}^{m+k,\, 2s+k-m} + w_{2s}^{2s+k-m,\, m+k}\right) - w_{2s}^{s+k,\, s+k} \right\} \end{aligned}$$

Therefore,

$$\begin{aligned}\|uv\|^2_{G^{p/2}_\sigma} &= \|A^{p/2}e^{\sigma A^{1/2}}(uv)\|^2_{L^2} \\ &= \sum_{s\geqslant 0}\lambda^p_{2s+1}e^{2\sigma\lambda^{1/2}_{2s+1}}\Bigg\|\sum_{k\geqslant 0}\sum_{m=0}^{s}\big(w^{m+k,\,2s+k+1-m}_{2s+1} \\ &\quad + w^{2s+k+1-m,\,m+k}_{2s+1}\big)\Bigg\|^2 \\ &\quad + \sum_{s\geqslant 0}\lambda^p_{2s}e^{2\sigma\lambda^{1/2}_{2s}}\Bigg\|\sum_{k\geqslant 0}\left\{\sum_{m=0}^{s}\big(w^{m+k,\,2s+k-m}_{2s}+w^{2s+k-m,\,m+k}_{2s}\big) \right.\\ &\quad \left. - w^{s+k,\,s+k}_{2s}\right\}\Bigg\|^2\end{aligned}$$

By using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\|uv\|^2_{G^{p/2}_\sigma} &\leqslant C\sum_{s\geqslant 0}\lambda^p_{2s}e^{2\sigma\lambda^{1/2}_{2s}}\left\{\sum_{k\geqslant 0}\sum_{m\geqslant 0}\lambda^{-(2p+1)/4}_{m+k}\right\} \\ &\quad\times\left\{\sum_{k\geqslant 0}\sum_{m=0}^{s}\|w^{m+k,\,2s+k-m}_{2s}+w^{2s+k-m,\,m+k}_{2s}\|^2\lambda^{(2p+1)/4}_{m+k}\right\} \\ &\quad + C\sum_{s\geqslant 0}\lambda^p_{2s+1}e^{2\sigma\lambda^{1/2}_{2s+1}}\left\{\sum_{k\geqslant 0}\sum_{m\geqslant 0}\lambda^{-(2p+1)/4}_{m+k}\right\} \\ &\quad\times\left\{\sum_{k\geqslant 0}\sum_{m=0}^{s}\|w^{m+k,\,2s+k+1-m}_{2s+1}+w^{2s+k+1-m,\,m+k}_{2s+1}\|^2\lambda^{(2p+1)/4}_{m+k}\right\}\end{aligned}$$

Since $p>3/2$, then $\sum_{k\geqslant 0}\sum_{m\geqslant 0}\lambda^{-(2p+1)/4}_{m+k}<\infty$. Also, the fact that $\lambda_n\leqslant\lambda_{n+k}$, for $n, k\geqslant 0$, yields

$$\begin{aligned}\|uv\|^2_{G^{p/2}_\sigma} &\leqslant C_p\sum_{s\geqslant 0}\sum_{k\geqslant 0}\lambda^p_{2s+2k}e^{2\sigma\lambda^{1/2}_{2s+2k}} \\ &\quad\times\left\{\sum_{m=0}^{s}\|w^{m+k,\,2s+k-m}_{2s}+w^{2s+k-m,\,m+k}_{2s}\|^2\lambda^{(2p+1)/4}_{m+k}\right\} \\ &\quad + C_p\sum_{s\geqslant 0}\sum_{k\geqslant 0}\lambda^p_{2s+2k+1}e^{2\sigma\lambda^{1/2}_{2s+2k+1}} \\ &\quad\times\left\{\sum_{m=0}^{s}\|w^{m+k,\,2s+k+1-m}_{2s+1}+w^{2s+k+1-m,\,m+k}_{2s+1}\|^2\lambda^{(2p+1)/4}_{m+k}\right\}\end{aligned}$$

Let $s' = s + k$, reset the summation $\sum_{s \geqslant 0} \sum_{k \geqslant 0} = \sum_{s' \geqslant 0} \sum_{k=0}^{k=s'}$, then replace s' by s and $m+k$ by m, and, finally, change the order of summations between the indices k and m; we obtain

$$\|uv\|^2_{G^{p/2}_\sigma} \leqslant C_p \sum_{s \geqslant 0} \sum_{m=0}^{s} \lambda^p_{2s} e^{2\sigma\lambda^{1/2}_{2s}} \left\{ \sum_{k=0}^{m} \|w^{m,\, 2s-m}_{2s-2k} + w^{2s-m,\, m}_{2s-2k}\|^2 \lambda_m^{(2p+1)/4} \right\}$$

$$+ C_p \sum_{s \geqslant 0} \sum_{m=0}^{s} \lambda^p_{2s+1} e^{2\sigma\lambda^{1/2}_{2s+1}} \left\{ \sum_{k=0}^{m} \|w^{m,\, 2s+1-m}_{2s+1-2k} \right.$$

$$\left. + w^{2s+1-m,\, m}_{2s+1-2k}\|^2 \lambda_m^{(2p+1)/4} \right\}$$

$$\leqslant C_p \sum_{s \geqslant 0} \lambda^p_{2s} e^{2\sigma\lambda^{1/2}_{2s}} \left\{ \sum_{m=0}^{s} \sum_{k=0}^{m} \|w^{m,\, 2s-m}_{2s-2k}\|^2 \lambda_m^{(2p+1)/4} \right.$$

$$\left. + \sum_{n=0}^{s} \sum_{k=0}^{n} \|w^{2s-n,\, n}_{2s-2k}\|^2 \lambda_n^{(2p+1)/4} \right\}$$

$$+ C_p \sum_{s \geqslant 0} \lambda^p_{2s+1} e^{2\sigma\lambda^{1/2}_{2s+1}} \left\{ \sum_{m=0}^{s} \sum_{k=0}^{m} \|w^{m,\, 2s+1-m}_{2s+1-2k}\|^2 \lambda_m^{(2p+1)/4} \right.$$

$$\left. + \sum_{n=0}^{s} \sum_{k=0}^{n} \|w^{2s+1-n,\, n}_{2s+1-2k}\|^2 \lambda_n^{(2p+1)/4} \right\}$$

$$\leqslant 2C_p \sum_{s \geqslant 0} \sum_{m+n=s} \sum_{k=0}^{\min\{m,n\}} \|w^{m,\, n}_{s-2k}\|^2 \lambda^{(2p+1)/4}_{\min\{m,n\}} \lambda^p_s e^{2\sigma\lambda^{1/2}_s}$$

It follows from (3.8) that

$$\|uv\|^2_{G^{p/2}_\sigma} \leqslant C_p \sum_{s \geqslant 0} \sum_{m+n=s} \|u_m v_n\|^2 \lambda^{(2p+1)/4}_{\min\{m,n\}} \lambda^p_s e^{2\sigma\lambda^{1/2}_s}$$

Finally, by using the fact that (cf. [8, 16])

$$\|u_m v_n\|^2 \leqslant C \min\{m, n\}\, \|u_m\|^2\, \|v_n\|^2 \tag{3.9}$$

and the convexity $\lambda_{j+\ell}^{1/2} \leqslant \lambda_j^{1/2} + \lambda_\ell^{1/2}$, we obtain

$$\begin{aligned}\|uv\|^2_{G_\sigma^{p/2}} &\leqslant C_p \sum_{s\geqslant 0} \sum_{m+n=s} \|u_m\|^2 \|v_n\|^2 \lambda_{\min\{m,n\}}^{(2p+3)/4} (\lambda_m^{1/2} + \lambda_n^{1/2})^{2p} e^{2\sigma(\lambda_m^{1/2}+\lambda_n^{1/2})} \\ &\leqslant 2^{2p} C_p \sum_{s\geqslant 0} \sum_{m+n=s} \|u_m\|^2 \|v_n\|^2 \lambda_{\min\{m,n\}}^{p} \lambda_{\max\{m,n\}}^{p} e^{2\sigma(\lambda_m^{1/2}+\lambda_n^{1/2})} \\ &\leqslant 2^{2p} C_p \sum_{s\geqslant 0} \sum_{m+n=s} \|u_m\|^2 \|v_n\|^2 \lambda_m^p \lambda_n^p e^{2\sigma(\lambda_m^{1/2}+\lambda_n^{1/2})} \\ &= 2^{2p} C_p \|u\|^2_{G_\sigma^{p/2}} \|v\|^2_{G_\sigma^{p/2}}\end{aligned}$$

which completes our proof.

Remark. In [6], the authors proved the same Lemma for periodic functions in the n-dimensional Euclidean space provided that $p > n/2$. In particular, the Lemma is true for $p > 1$ in the case of periodic functions in the two dimensional Euclidean space. Here, we proved the Lemma for $p > 3/2$ on the two dimensional sphere. The reason for this stronger restriction on p is due to the fact that in the case of periodic boundary conditions the eigenfunctions of the operator A satisfy $\phi_n \phi_m = \phi_{m+n}$, while in the case of spherical harmonics we have the more involved identity (3.7). However, we believe that Lemma 1 should hold for $p > 1$, because the proof of Lemma 1 should be related to the proof of the fact that the Sobolev space $H^p(S)$ is a topological algebra for $p > 1$.

Lemma 2. *Let $u \in G_\sigma^{p/2}$ and let p, σ be as in Lemma 1, and f_i, g_i be as in (2.1) and (2.2), for $i = 1, 2, 3, 4$. Then $f_i(u) \in G_\sigma^{p/2}$ and*

$$\|f_i(u)\|_{G_\sigma^{p/2}} \leqslant g_i(c_p \|u\|_{G_\sigma^{p/2}}) \qquad \text{for} \quad i = 1, 2, 3, 4 \tag{3.10}$$

Proof. Let $1 \leqslant i \leqslant 4$ and assume that $f_i(s) = \sum_{j=0}^{\infty} a_{ij} s^j$ for all $s \in \mathbb{R}$. Then it follows from Lemma 1 that

$$\begin{aligned}\left\| \sum_{j=0}^{n} a_{ij} u^j \right\|_{G_\sigma^{p/2}} &\leqslant \sum_{j=0}^{n} |a_{ij}| \, \|u^j\|_{G_\sigma^{p/2}} \\ &\leqslant \sum_{j=0}^{n} |a_{ij}| \, (c_p \|u\|_{G_\sigma^{p/2}})^j \leqslant g_i(c_p \|u\|_{G_\sigma^{p/2}}) \end{aligned} \tag{3.11}$$

Also, it follows from (3.2) that

$$\left\| \sum_{j=m}^{n} a_{ij} u^j \right\|_{G_\sigma^{p/2}} \leqslant \sum_{j=m}^{n} |a_{ij}| \, (c_p \|u\|_{G_\sigma^{p/2}})^j \to 0 \qquad \text{as} \quad m, n \to \infty$$

Therefore, $\sum_{j=0}^{n} a_{ij} u^j$ converges in $G_\sigma^{p/2}$ as $n \to \infty$; say

$$\sum_{j=0}^{n} a_{ij} u^j \to \tilde{f}_i(\cdot) \qquad \text{in } G_\sigma^{p/2}, \qquad \text{as} \quad n \to \infty \tag{3.12}$$

Now, since $G_\sigma^{p/2} \hookrightarrow H^p \hookrightarrow C^0(S)$ for $p > 1$, where $C^0(S)$ is the space of continuous function on S, then it follows that $\sum_{j=1}^{n} a_{ij} u(\cdot)^j$ converges uniformly on S. Since $\sum_{j=1}^{\infty} a_{ij} u(s)^j = f_i(u(s))$ for each $s \in S$, then $\sum_{j=1}^{n} a_{ij} u(\cdot)^j \to f_i(u(\cdot))$ uniformly on S. Hence, $\tilde{f}_i(s) = f_i(u(s))$ for all $s \in S$. Finally, by passing to the limit on the left-hand side of (2.11), we obtain

$$\|f_i(u)\|_{G_\sigma^{p/2}} \leqslant g_i \left(c_p \, \|u\|_{G_\sigma^{p/2}}\right) \tag{3.13}$$

Proof of Theorem 1. We first prove the uniqueness of regular solutions following closely the argument given in Chapter III of [20] for the Navier–Stokes equations. Assume that u is a regular solution on $[0, T]$. Since $H^1(S) \subseteq L^2(S) \subseteq (H^1(S))'$, from (3.5) we have

$$\frac{d}{dt} \langle u, \phi \rangle = -\langle \nu A u + \tilde{F}(u), \phi \rangle$$

for almost all $t \in [0, T]$, for every $\phi \in H^1(S)$, where $\langle \, , \rangle$ is the scalar product in duality between $H^1(S)$ and $(H^1(S))'$, and $\tilde{F}(u) := F(u, \nabla u)$. Next, by the regularity of u and $\nu A u + \tilde{F}(u) \in L^1([0, T]; (H^1(S))')$, and thus by Lemma 1.1 of Chapter III in [20], we have that

$$\frac{du}{dt} = -\nu A u - \tilde{F}(u) \qquad \text{in } (H^1(S))', \qquad \text{a.e. in } [0, T] \tag{3.14}$$

and, moreover, that u is almost everywhere equal to a continuous function from $[0, T]$ into $(H^1(S))'$, so that the initial condition is well defined.

Now assume that u and v are two regular solutions of (2.1) and (2.2) on the interval $[0, T]$ with the same initial value u_0; from (3.14) we have

$$\frac{d(u-v)}{dt} = -\nu A(u-v) - (\tilde{F}(u) - \tilde{F}(v)) \qquad \text{in } (H^1(S))' \tag{3.15}$$

almost everywhere in $[0, T]$. Since $(u-v)(t) \in H^1(S)$ for almost every $t \in [0, T]$, we can take the scalar product of Eq. (3.15) almost everywhere in time in the duality between $H^1(S)$ and $(H^1(S))'$ to get that

$$\left\langle \frac{d(u-v)}{dt}, u-v \right\rangle = -\langle \nu A(u-v), u-v \rangle - \langle \tilde{F}(u) - \tilde{F}(v), u-v \rangle$$

holds almost everywhere in $[0, T]$. Since for any $\phi \in L^2([0, T]; H^1(S))$ with $d\phi/dt \in L^2([0, T]; (H^1(S))')$, then Lemma 1.2 of Chapter III in [20] yields

$$2\left\langle \frac{d\phi(t)}{dt}, \phi(t) \right\rangle = \frac{d}{dt} \|\phi(t)\|_{L^2}^2$$

therefore we have for $\phi = u - v$,

$$\frac{1}{2}\frac{d}{dt} \|u - v\|_{L^2}^2 + \nu \|A^{1/2}(u - v)\|_{L^2}^2 = -\langle \tilde{F}(u) - \tilde{F}(v), u - v\rangle \tag{3.16}$$

Since $u, v \in C([0, T]; H^p)$ and $H^p \hookrightarrow C^0(S)$ for $p > 1$, we have that $u, v \in C([0, T]; L^\infty)$. Let $M = 2\max\{\sup_{0 \leqslant t \leqslant T} \|u(\cdot, t)\|_\infty, \sup_{0 \leqslant t \leqslant T} \|v(\cdot, t)\|_\infty\}$.

In the case when the nonlinearity is independent of ∇u, i.e., when $a = 0$, we have

$$\begin{aligned} -\langle \tilde{F}(u) - \tilde{F}(v), u - v\rangle &\leqslant \|f_4(u) - f_4(v)\|_{L^2} \|u - v\|_{L^2} \\ &\leqslant \sup_{|z| \leqslant M} |f_4'(z)| \|u - v\|_{L^2}^2 \\ &\leqslant C \|u - v\|_{L^2}^2 \end{aligned} \tag{3.17}$$

In this case, we deduce from (3.16) and (3.17) that

$$\frac{d}{dt} \|u(t) - v(t)\|_{L^2}^2 \leqslant C \|u(t) - v(t)\|_{L^2}^2 \tag{3.18}$$

which implies that $\|u(t) - v(t)\|_{L^2}^2 = 0$ for almost all $t \in [0, T]$, because $u(0) = v(0)$.

We now consider the case when $a = 1$. Here, we have

$$\begin{aligned} &\langle \tilde{F}(u) - \tilde{F}(v), u - v\rangle \\ &\quad = \langle (\vec{f}(u) - \vec{f}(v)) \cdot \nabla u, u - v\rangle \\ &\qquad + \langle \vec{f}(v) \cdot (\nabla u - \nabla v), u - v\rangle + \langle f_4(u) - f_4(v), u - v\rangle \\ &\quad = J_1 + J_2 + J_3 \end{aligned} \tag{3.19}$$

Thanks to the Cauchy–Schwarz inequality, we have

$$J_2 \leqslant C(\sup_{|z| \leqslant M} |f_1(z)|) \|A^{1/2}(u - v)\|_{L^2} \|u - v\|_{L^2} \tag{3.20}$$

On the other hand, Hölder's inequality yields

$$J_1 \leqslant C \, \||\nabla u\||_{L^2} \, \|u-v\|_{L^4} \, \|\vec{f}(u)-\vec{f}(v)\|_{L^4}$$

By using the mean value theorem, one has

$$J_1 \leqslant C \, (\sup_{|z| \leqslant M} \, |\vec{f}'(z)|) \, \||\nabla u\||_{L^2} \, \|u-v\|_{L^4}^2$$

By using the interpolation inequality (cf. [2]), $\|\phi\|_{L^4} \leqslant \|\phi\|_{L^2}^{1/2} \, \|A^{1/2}\phi\|_{L^2}^{1/2}$, it follows that

$$\begin{aligned} J_1 &\leqslant C(M) \, \||\nabla u\||_{L^2} \, \|u-v\|_{L^2} \, \|A^{1/2}(u-v)\|_{L^2} \\ &\leqslant C(M) \, \|u\|_{H^p} \, \|u-v\|_{L^2} \, \|A^{1/2}(u-v)\|_{L^2} \end{aligned} \tag{3.21}$$

where $C(M) = C(\sup_{|z| \leqslant M} |\vec{f}'(z)|)$. From (3.19)–(3.21), we have

$$-\langle \tilde{F}(u) - \tilde{F}(v), u-v \rangle \leqslant \frac{K(M)}{\nu} \{ \|u\|_{H^p}^2 + 1 \} \, \|u-v\|_{L^2}^2 + \frac{\nu}{2} \, \|A^{1/2}(u-v)\|_{L^2}^2 \tag{3.22}$$

where $K(M)$ is a constant that depends on M. Now (3.18) and (3.22) yield

$$\frac{d}{dt} \|u(t)-v(t)\|_{L^2}^2 \leqslant \frac{K(M)}{\nu} [\, \|u(\tau)\|_{H^p}^2 + 1 \,] \, \|u(t)-v(t)\|_{L^2}^2 \tag{3.23}$$

By Gronwall's inequality and the fact that $\sup_{0 \leqslant t \leqslant T} \|u(t)\|_{H^p} < \infty$, we have

$$\|u(t)-v(t)\|_{L^2}^2 = 0$$

for almost every $t \in [0, T]$. So, the uniqueness of regular solutions is established.

To prove the existence of a regular solution, we use the standard Galerkin approximations and energy estimates. Let P^N be the orthogonal projection of $L^2(S)$ onto H_N, where H_N is the linear space spanned by the set

$$\{ Y_n^m : m = 0, \pm 1, ..., \pm n; n = 0, 1, 2, ..., N \}$$

Let

$$u^N(x, t) = \sum_{n \leqslant N, m} \alpha_n^m(N, t) \, Y_n^m \equiv \sum_{n=0}^{N} \sum_{-n \leqslant m \leqslant n} \alpha_n^m(N, t) \, Y_n^m$$

be the solution of the Galerkin system

$$\frac{d}{dt} u^N + \nu A u^N + P^N \widetilde{F}(u^N) = 0 \tag{3.24}$$

$$u^N(0) = P^N u^0 \tag{3.25}$$

For each N, it is easy to see that solving (3.24) and (3.25) reduces to solving a nonlinear system of first-order ordinary differential equations for the Fourier coefficients $\alpha_n^m(N, t)$'s. This system has nonlinearities that are Lipschitz continuous and with initial conditions given by the Fourier coefficients of $u^N(0) = P^N u^0$. We deduce that (3.24) and (3.25) have a unique solution $u^N(\cdot, t)$ on some interval $[0, \widetilde{T}_N]$. Moreover, $u^N(\cdot, t) \in G_\sigma^{p/2}$ for all p and σ.

Next, we derive an a priori estimate for u^N as follows. Apply $A^{p/2} e^{tA^{1/2}}$ to (2.24) and take the L^2-inner product with $A^{p/2} e^{tA^{1/2}} u^N$; we obtain

$$\left\langle A^{p/2} e^{tA^{1/2}} \frac{d}{dt} u^N, A^{p/2} e^{tA^{1/2}} u^N \right\rangle_{L^2} + \nu \langle A^{p/2} e^{tA^{1/2}} A u^N, A^{p/2} e^{tA^{12}} u^N \rangle_{L^2}$$
$$+ \langle A^{p/2} e^{tA^{1/2}} P^N \widetilde{F}(u^N), A^{p/2} e^{tA^{1/2}} u^N \rangle_{L^2} = 0$$

By the symmetry of $A^{1/2}$, one has

$$\frac{1}{2} \frac{d}{dt} \| A^{p/2} e^{tA^{1/2}} u^N \|_{L^2}^2 - \langle A^{(p+1)/2} e^{tA^{1/2}} u^N, A^{p/2} e^{tA^{1/2}} u^N \rangle_{L^2}$$
$$+ \nu \| A^{(p+1)/2} e^{tA^{1/2}} u^N \|_{L^2}^2$$
$$= - \langle A^{p/2} e^{tA^{1/2}} P^N \widetilde{F}(u^N), A^{p/2} e^{tA^{12}} u^N \rangle_{L^2} \tag{3.26}$$

By Young's inequality, we have

$$\frac{1}{2} \frac{d}{dt} \| A^{p/2} e^{tA^{1/2}} u^N \|_{L^2}^2 + \frac{\nu}{2} \| A^{(p+1)/2} e^{tA^{1/2}} u^N \|_{L^2}^2 - \frac{1}{2\nu} \| A^{p/2} e^{tA^{1/2}} u^N \|_{L^2}^2$$
$$\leqslant - \langle A^{p/2} e^{tA^{1/2}} P^N \widetilde{F}(u^N), A^{p/2} e^{tA^{1/2}} u^N \rangle_{L^2} \tag{3.27}$$

Thanks to Lemma 1 and Lemma 2 and the fact that

$$\| P^N \widetilde{F}(u^N) \|_{G_t^{p/2}} \leqslant \| \widetilde{F}(u^N) \|_{G_t^{p/2}}$$

we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^N(t)\|^2_{G_t^{p/2}}+\frac{\nu}{2}\|u^N(t)\|^2_{G_t^{(p+1)/2}}\\
&\quad\leqslant\frac{1}{2\nu}\|u^N(t)\|^2_{G_t^{p/2}}+\|P^N\widetilde{F}(u^N)\|_{G_t^{p/2}}\|u^N\|_{G_t^{p/2}}\\
&\quad\leqslant\frac{1}{2\nu}\|u^N(t)\|^2_{G_t^{p/2}}+\|\vec{f}(u^N)\cdot\nabla u^N)^a+f_4(u^N)\|_{G_t^{p/2}}\|u^N\|_{G_t^{p/2}}\\
&\quad\leqslant\frac{1}{2\nu}\|u^N(t)\|^2_{G_t^{p/2}}+g(c_p\|u^N\|_{G_t^{p/2}})\|u^N\|_{G_t^{p/2}}\|u^N\|^a_{G_t^{(p+1)/2}}
\end{aligned}\tag{3.28}$$

where $a=0$ or 1, and $g(c_p\|u^N\|_{G_t^{p/2}})=\max_{1\leqslant i\leqslant 4} g_i(c_p\|u^N\|_{G_t^{p/2}})$.

Now if $a=1$, then by Young's inequality we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^N(t)\|^2_{G_t^{p/2}}+\frac{\nu}{2}\|u^N(t)\|^2_{G_t^{(p+1)/2}}\\
&\quad\leqslant\frac{1}{2\nu}\|u^N(t)\|^2_{G_t^{p/2}}+\frac{1}{\nu}(g(c_p\|u^N\|_{G_t^{p/2}}))^2\|u^N\|^2_{G_t^{p/2}}+\frac{\nu}{4}\|u^N\|^2_{G_t^{(p+1)/2}}
\end{aligned}\tag{3.29}$$

which leads to

$$\begin{aligned}
&\frac{d}{dt}\|u^N(t)\|^2_{G_t^{p/2}}+\frac{\nu}{2}\|u^N(t)\|^2_{G_t^{(p+1)/2}}\\
&\quad\leqslant\frac{1}{\nu}\|u^N(t)\|^2_{G_t^{p/2}}[1+2(g(c_p\|u^N\|_{G_t^{p/2}}))^2]\\
&\quad:=\frac{1}{\nu}\|u^N(t)\|_{G_t^{p/2}}h_1(\|u^N(t)\|_{G_t^{p/2}})
\end{aligned}\tag{3.30}$$

where h_1 is the function given by

$$h_1(s)=s[1+2(g(c_ps))^2]\tag{3.31}$$

Now, if $a=0$, then (3.28) will be replaced by

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|u^N(t)\|^2_{G_t^{p/2}}+\frac{\nu}{2}\|u^N(t)\|^2_{G_t^{(p+1)/2}}\\
&\quad\leqslant\frac{1}{2\nu}\|u^N(t)\|^2_{G_t^{p/2}}+g(c_p\|u^N(t)\|_{G_t^{p/2}})\|u^N(t)\|_{G_t^{p/2}}
\end{aligned}\tag{3.32}$$

Therefore, in this case we have

$$\frac{d}{dt} \|u^N(t)\|^2_{G_t^{p/2}} + \nu \|u^N(t)\|^2_{G_t^{(p+1)/2}}$$

$$\leqslant \frac{1}{\nu} \|u^N(t)\|_{G_t^{p/2}} h_0(\|u^N(t)\|_{G_t^{p/2}}) \tag{3.33}$$

where

$$h_0(s) := s + 2\nu g(c_p s) \tag{3.34}$$

Thus, in either case, $a=0$ or $a=1$; we deduce from (3.30) and (3.33) that there exists a real analytic function $h_a(s)$ such that $h_a(s) = \sum_{j=0}^{\infty} |c_{a,j}| s^j$, which converges for all $s \in \mathbb{R}$, and such that

$$\|u^N(t)\|_{G_t^{p/2}} + \frac{\nu}{2} \int_0^t \|u^N(s)\|^2_{G_s^{(p+1)/2}} ds$$

$$\leqslant \|A^{p/2} u^N(0)\|_{L^2} + \frac{1}{\nu} \int_0^t h_a(\|u^N(s)\|_{G_s^{p/2}}) ds \tag{3.35}$$

Since $\|u^N(t)\|_{G_t^{p/2}}$ is continuous in t and $\|A^{p/2} u^N(0)\|_{L^2} = \|u^N(0)\|_{G_0^{p/2}} \leqslant \|u^0\|_{H^p} \leqslant M_0$, then there exists a constant $T_N^* > 0$ such that

$$\|u^N(t)\|_{G_t^{p/2}} \leqslant 1 + 2 \|u^N(0)\|_{G_0^{p/2}} \leqslant 1 + 2 \|u^0\|_{H^p} \leqslant 1 + 2M_0 \tag{3.36}$$

for $t \in [0, T_N^*]$. Since $h_a(s)$ is an increasing function, for $s \geqslant 0$, then it follows from (2.35) and (2.36) that

$$\|u^N(t)\|_{G_t^{p/2}} \leqslant \|u^0\|_{H^p} + \frac{t}{\nu} h_a(1 + 2 \|u^0\|_{H^p}) \tag{3.37}$$

for $0 \leqslant t \leqslant T_N^*$. Now choose T^* (T^* will depend on a) such that

$$\|u^0\|_{H^p} + \frac{T^*}{\nu} h_a(1 + 2 \|u^0\|_{H^p}) = 1 + 2 \|u^0\|_{H^p} \tag{3.38}$$

i.e.,

$$T^* = \nu(1 + \|u^0\|_{H^p})(h_a(1 + 2 \|u^0\|_{H^p}))^{-1} > 0 \tag{3.39}$$

We deduce from (3.37) and (3.38) that

$$\|u^N(t)\|_{G_t^{p/2}} \leqslant 1 + 2 \|u^0\|_{H^p} \leqslant 1 + 2M_0 \tag{3.40}$$

for $0 \leqslant t \leqslant T^*$. Moreover, we have

$$\sup_{0 \leqslant t \leqslant T^*} \|u^N(t)\|_{H^p} \leqslant \sup_{0 \leqslant t \leqslant T^*} \|u^N(t)\|_{G_t^{p/2}} \leqslant 1 + 2M_0 \tag{3.41}$$

for all $N \geqslant 1$. Since $H^p \hookrightarrow C^0(S)$ for $p > 1$, then we have

$$\sup_{0 \leqslant t \leqslant T^*} \|u^N(t)\|_\infty \leqslant C \qquad \text{for all} \quad N \geqslant 1 \tag{3.42}$$

It also follows from (3.35) and (3.38) that

$$\nu \int_0^{T^*} \|A^{(p+1)/2} u^N(s)\|_{L^2}^2 \, ds \leqslant \nu \int_0^{T^*} \|u^N(s)\|_{G_t^{(p+1)/2}}^2 \, ds \leqslant C \tag{3.43}$$

Furthermore, (3.24), (3.41), (3.42), and (3.43) yield the estimate

$$\int_0^{T^*} \left\| \frac{d}{ds} u^N(s) \right\|_{L^2}^2 ds \leqslant \text{constant} \equiv C \tag{3.44}$$

By using the above estimates and Aubin's compactness theorem (see [2, 3, 14, 21]), we conclude that there exists a subsequence of $\{u^N\}$, which we still denote $\{u^N\}$, and a function u such that

$$\begin{aligned} u^N &\to u \qquad \text{strongly in } L^2(0, T^*; H^p), \\ u^N &\to u \qquad \text{weakly in } L^2(0, T^*; H^{p+1}), \\ u &\in L^\infty(0, T^*; H^p) \end{aligned}$$

and

$$\sup_{0 \leqslant t \leqslant T^*} \|u(t)\|_\infty \leqslant \text{constant} \equiv C$$

It remains to show that the limit function u satisfies (3.5). Let $\phi \in H^1$ be arbitrary. Then it follows from (3.24) that

$$\langle u^N(t), \phi \rangle_{L^2} - \langle u^N(t_0), \phi \rangle_{L^2} + \nu \int_{t_0}^t \langle A^{1/2} u^N(\tau), A^{1/2} \phi \rangle_{L^2} \, d\tau$$

$$+ \int_{t_0}^t \langle P^N \tilde{F}(u^N(\tau)), \phi \rangle_{L^2} \, d\tau = 0 \tag{3.45}$$

Based on the properties of the sequence $\{u^N\}$ above, we can extract a subsequence, still denoted $\{u^N\}$, such that $u^N(t) \to u(t)$ strongly in H^p for almost all $t \in [0, T^*]$, and $u^N(t) \to u(t)$ weakly in H^{p+1} for almost all $t \in [0, T^*]$. For this subsequence, it is clear that the first three terms in (3.45) converge to

$$\langle u(t), \phi \rangle_{L^2} - \langle u(t_0), \phi \rangle_{L^2} + \nu \int_{t_0}^{t} \langle A^{1/2} u(\tau), A^{1/2}\phi \rangle_{L^2} \, d\tau \tag{3.46}$$

for almost all $t, t_0 \in [0, T^*]$. For the fourth term in (3.45), we first consider the case when $a = 0$. In this case we have

$$\begin{aligned}
&\|P^N \widetilde{F}(u^N(\tau)) - \widetilde{F}(u(\tau))\|_{L^2}^2 \\
&\quad \leqslant 2\, \|P^N f_4(u^N(\tau)) - P^N f_4(u(\tau))\|_{L^2}^2 + 2\, \|P^N f_4(u(\tau)) - f_4(u(\tau))\|_{L^2}^2 \\
&\quad \leqslant 2 \sup_{|z| \leqslant C} |f'_4(z)|^2 \, \|u^N(\tau) - u(\tau)\|_{L^2}^2 + 2\, \|P^N f_4(u(\tau)) - f_4(u(\tau))\|_{L^2}^2
\end{aligned} \tag{3.47}$$

which converges to zero as $N \to \infty$, for almost all $\tau \in [0, T^*]$.

For the case $a = 1$, one has

$$\begin{aligned}
&\|P^N(\vec{f}(u^N(\tau)) \cdot \nabla u^N(\tau)) - \vec{f}(u(\tau)) \cdot \nabla u(\tau)\|_{L^2}^2 \\
&\quad \leqslant 2\, \|\vec{f}(u^N(\tau)) \cdot (\nabla u^N(\tau) - \nabla u(\tau))\|_{L^2}^2 \\
&\qquad + 2\, \|(\vec{f}(u^N(\tau)) - \vec{f}(u(\tau))) \cdot \nabla u(\tau)\|_{L^2}^2 \\
&\qquad + 2\, \|P^N(\vec{f}(u(\tau)) \cdot \nabla u(\tau)) - \vec{f}(u(\tau)) \cdot \nabla u(\tau))\|_{L^2}^2 \\
&\quad \leqslant C\, \|\nabla u^N(\tau)\|_{\infty}^2 \, (\sup_{|z| \leqslant C,\, 1 \leqslant i \leqslant 4} |f'_i(z)|)^2 \, \|u^N(\tau) - u(\tau)\|_{L^2}^2 \\
&\qquad + C \sup_{|z| \leqslant C,\, 1 \leqslant i \leqslant 4} |f_i(z)|^2 \, \|A^{1/2}(u^N(\tau) - u(\tau))\|_{L^2}^2 \\
&\qquad + C\, \|P^N(\vec{f}(u) \cdot \nabla u) - \vec{f}(u) \cdot \nabla u\|_{L^2}^2
\end{aligned} \tag{3.48}$$

Since $\|\nabla u^N(\tau)\|_{\infty} \leqslant c\, \|u^N(\tau)\|_{H^{p+1}}$ and $u^N(\tau) \to u(\tau)$ weakly in H^{p+1} for almost every $\tau \in [0, T^*]$, then $\|\nabla u^N(\tau)\|_{\infty}$ is bounded for almost every $\tau \in [0, T^*]$. We conclude that the right-hand side of (3.48) tends to 0 as $N \to \infty$, for almost every $\tau \in [0, T^*]$. From (3.47) and (3.48), we obtain that $\|P^N \widetilde{F}(u^N(\tau)) - \widetilde{F}(u(\tau))\|_{L^2}$ converges to zero as $N \to \infty$, for almost all $\tau \in [0, T^*]$. Hence, u satisfies (3.5). By the uniqueness of regular solutions and by estimate (3.41), it follows that $u(t) \in G_t^{p/2}$ for all $t \in [0, T^*]$.

Corollary 1. *If* $\|u(t)\|_{H^p} \leqslant M_0$ *for all* $t > 0$ *and* $p > 3/2$, *then* $u(t) \in G_t^{p/2}$ *for all* $t \in [0, T^*]$ *and* $u(t) \in G_{T^*}^{p/2}$ *for all* $t \geqslant T^*$.

Proof. We need to prove only that $u(T^* + t) \in G_{T^*}^{p/2}$ for $t \in [0, T^*]$, and the statement of the Corollary will follow by repeating the argument. Define $v(s) = u(s + t)$; $0 \leqslant s, t \leqslant T^*$. By assumption, we have $\|v(0)\|_{H^p} = \|v(t)\|_{H^p} \leqslant M_0$. So, by applying Theorem 1, we deduce that $v(s) = u(s + t) \in G_s^{p/2}$ for $s \in [0, T^*]$. Therefore, $v(T^*) = u(T^* + t) \in G_{T^*}^{p/2}$ for $t \in [0, T^*]$.

3.2. General Nonlinearity

In the previous section, we assumed that the initial data $u_0 \in H^p$ for $\frac{3}{2} < p \leqslant \frac{5}{2}$, and because of this we had to impose some restrictions on the structure of the nonlinearity as in (2.1) with $a = 0$ or 1. However, if the initial data are smoother, i.e., $u^0 \in H^p(S)$ with $p > 5/2$, then there will be no restriction on the structure of the nonlinear term $F(u, \nabla u)$ as long as $F(u, \nabla u)$ is analytic with respect to all its variables.

Theorem 2. *Let* $M_0 > 0$ *be given and let* $u^0 \in H^p$ *with* $p > 5/2$ *such that* $\|u^0\|_{H^p} \leqslant M_0$. *Then there exists a constant* $T^* > 0$ *depending only on* M_0 *and* ν, *such that (1.3) and (1.4) have a unique regular solution* u *on* $[0, T^*]$. *Moreover,* $u(t) \in G_t^{p/2}$ *for all* $t \in [0, T^*]$.

Proof. The proof is somewhat similar to the proof of Theorem 1. We refer the reader to [6] for the details of the proof.

4. REMARKS

1. In this paper, we deal with analytic nonlinear parabolic equations on the two-dimensional sphere. The same proof works for the same type of equations on the n-dimensional sphere provided that $p > (n+1)/2$. However, here again, we believe that one should be able to prove this general result for $p > n/2$.

2. Following the proofs presented in [4, 5, 11, and 12], one can use the Gevrey regularity results to demonstrate the exponentially fast convergence, in a certain sense, of the Galerkin approximation, based on the spherical harmonics, to the unique exact regular solution of the underlying equation on the sphere. This result also indicates that the Galerkin method, based on the spherical harmonics, is a very reliable numerical method in this case.

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