ASYMPTOTIC BEHAVIOR OF VISCOUS 1–D SCALAR CONSERVATION LAWS WITH NEUMANN BOUNDARY CONDITIONS

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ABSTRACT. In this paper we consider the long–time behavior of a generalized viscous Burgers equation – a one dimensional scalar conservation law with viscosity – subject to Neumann boundary conditions. We show that all the steady state solutions of this problem are constant functions. Furthermore, we prove that, for any initial data, the time dependent solution converges to a steady state solution, as the time grows unboundedly to infinity.

1. INTRODUCTION

Starting from the pioneer work of Burgers [3], [17] and Hopf [9] the Burgers equation has always been used as a paradigm for shedding light on understanding turbulence and other nonlinear phenomena (see, for example, [12], [13], [19], [20], [22], [23] and references theirin). In recent years, Burgers equation has also been used as a model for studying boundary and distributed parameter feedback control of nonlinear partial differential equations (see, for example, [2], [4], [5], [6], [10], [14], [15], [16] and references theirin).

In this paper, we study and characterize the long–time behavior of solutions to the unforced viscous Burgers equation subject to Neumann boundary conditions. This problem was brought to our attention by Professor D.S. Gilliam [15]. The tools and results we present here are equally valid to a larger class of viscous one dimensional scalar conservation laws, of which the Burgers equation is a special case. Therefore, we will study here the long–time behavior of the following class of one dimensional generalized Burgers equation in the interval $\Omega = (0,1)$, subject to the homogeneous (no flux) Neumann boundary conditions,

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}(F(u)) = 0, \qquad \text{for } x \in \Omega,\ t > 0, \tag{1}$$

$$u_x(0,t) = u_x(1,t) = 0, \tag{2}$$

$$u(x,0) = u_{\text{in}}(x), \qquad \text{for } x \in \Omega, \tag{3}$$

where $u_{\text{in}}(x) \in C^0(\overline{\Omega})$, the initial data, is given. The function F is assumed to be in $C^2(\mathbb{R})$ with $F'' \geq 0$. Notice that in the case of Burgers equation $F(u) = \frac{1}{2}u^2$.

It is well known that in the case of Dirichlet boundary conditions the viscous Burgers equation, with a source term, has a unique steady state which attracts all the time dependent solutions. That is in the case of Dirichlet boundary conditions the global attractor consists of a single stable steady state (see, for example, [8]

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and [13]). However, in the case of the Neumann boundary conditions, the steady states are not unique. Every constant function is a steady state solution. Thus, the long–time behavior in this case is not necessary trivial. Nonetheless, we will show, in section 3, that each time dependent solution of the above system (1)–(3) converges to a constant function, hence to a steady state solution. We are still unable, however, to find a simple connection between the asymptotic limiting constant function and the initial data. This interesting problem is a subject of future analytical and numerical work.

After the completion of an earlier draft of this paper it was brought to our attention that in [24] the author proved the same results for a more general class of 1–D second order parabolic equations, including the system (1)–(3), by showing that there are nontrivial local *Liapunov Functionals* for those systems. We refer the reader to [25] for details. However, our approach is different and much more elementary. We rely substantially on the *Maximum Principle.* In the *Liapunov's Functional* context of [24] we prove that

$$\int_0^1 |w_x|^2 \, dx$$

is a *Liapunov's Functional*, where w is an auxiliary function we introduce later in (65).

2. Global Existence and Regularity

In this section we introduce some notions and establish the global existence and uniqueness of the strong solution to the initial Neumann boundary value problem (1)–(3). In particular, we prove the *maximum principle* which plays an essential role in our analysis. We provide detailed proofs of these results for the sake of completeness. Moreover, as far as we are aware, such details for weak solutions of the Neumann boundary value problem are not available in literature (see, however, [18] for the case of classical solutions).

Denote by

$$\mathcal{V} \quad = \quad \{u \in C^\infty(\overline{\Omega}) : \quad u_x(0) = u_x(1) = 0\},$$

and

$$\widetilde{F}(a) = \max_{|x|\leq a}\{|F(x)|, |F'(x)|, |F''(x)|\}. \tag{4}$$

Definition 1. *Let* $u_{\text{in}} \in C^0(\overline{\Omega})$ *and let* T *be any positive number. A function* $u(x,t)$ *is called a regular solution of* (1)–(3) *on* $[0,T]$ *if*

$$u \in L^2_{\text{loc}}((0,T]; H^3(\Omega)) \cap C((0,T]; H^2(\Omega)) \cap L^2([0,T]; H^1(\Omega)) \cap C([0,T]; L^2(\Omega))$$

and if it satisfies

$$\langle u(t), \phi\rangle_{L^2} - \langle u(t_0), \phi\rangle_{L^2} + \int_{t_0}^{t} \langle u_x(s), \phi_x\rangle_{L^2}\, ds + \tag{5}$$

$$+ \int_{t_0}^{t} \langle (F(u(s))_x, \phi\rangle_{L^2}\, ds = 0,$$

for every $\phi \in \mathcal{V}$ *and every* t, $t_0 \in [0,T]$.

Proposition 2. (maximum principle)
Let $u(x,t)$ be a regular solution of the system (1)–(3) in the interval $[0,T]$. Then,

$$\max_{0\le x\le 1} u(x,t) \le \max_{0\le x\le 1} u(x,t_0) \tag{6}$$

$$\min_{0\le x\le 1} u(x,t) \ge \min_{0\le x\le 1} u(x,t_0) \tag{7}$$

for every $0 \le t_0 \le t \le T$.

Proof. Let $t_0 \in (0,T]$ be fixed and $u(x,t)$ be a regular solution on $[0,T]$. First let us observe that from the definition of a regular solution that $u, u_x \in C^0((0,T]\times\overline{\Omega})$. As a result

$$F(u(x,t)) \in C^0((0,T]\times\overline{\Omega})$$

and

$$(F(u(x,t)))_x = F'(u(x,t))u_x(x,t) \in C^0((0,T]\times\overline{\Omega}).$$

Moreover, we also have

$$u_{xx} - F'(u)u_x \ \in C^0((0,T],L^2(\Omega)).$$

Therefore, we get

$$u_t = u_{xx} - F'(u)u_x \ \in C^0((0,T],L^2(\Omega)).$$

Finally, it is clear that since $u \in L^2_{\text{loc}}((0,T];H^3(\Omega))$ we obtain

$$\left|\int_{t_0}^t u_{xx}(x,s)\,ds - \int_{t_0}^\tau u_{xx}(y,s)\,ds\right| \le \left|\int_\tau^t u_{xx}(x,s)\,ds\right| + \int_{t_0}^t |u_{xx}(x,s) - u_{xx}(y,s)|\,ds.$$

By Cauchy–Schwarz inequality, we have

$$\left|\int_\tau^t u_{xx}(x,s)\,ds\right| \le |t-\tau|^{\frac12}\left(\int_\tau^t \|u_{xx}(\cdot,s)\|^2_{L^\infty}\,ds\right)^{\frac12},$$

and

$$|u_{xx}(x,s) - u_{xx}(y,s)| = \left|\int_y^x u_{xxx}(\xi,s)\,d\xi\right| \le C\,\|u(\cdot,s)\|_{H^3(\Omega)}\,ds\,|x-y|^{\frac12}.$$

Note that

$$\|u_{xx}(\cdot,s)\|^2_{L^\infty} \le C\|u_{xx}(\cdot,s)\|_{L^2(\Omega)}\|u(\cdot,s)\|_{H^3(\Omega)} \le C\|u(\cdot,s)\|^2_{H^3(\Omega)}.$$

Thus

$$\left|\int_\tau^t u_{xx}(x,s)\,ds\right| \le C\,|t-\tau|^{\frac12}\left(\int_\tau^t \|u(\cdot,s)\|^2_{H^3(\Omega)}\,ds\right)^{\frac12}.$$

Therefore, we get

$$\left|\int_{t_0}^t u_{xx}(x,s)\,ds - \int_{t_0}^\tau u_{xx}(y,s)\,ds\right| \le C\,|t-\tau|^{\frac12}\left(\int_\tau^t \|u(\cdot,s)\|^2_{H^3(\Omega)}\,ds\right)^{\frac12} + C\int_{t_0}^t \|u(\cdot,s)\|_{H^3(\Omega)}\,ds\,|x-y|^{\frac12}.$$

Thus,

$$\int_{t_0}^t u_{xx}(x,s)\,ds \in C^0((0,T]\times\overline{\Omega}),$$

as a function of x and t. Combining the above we conclude that

$$u(x,t) - u(x,t_0) - \int_{t_0}^{t} u_{xx}(x,s)\,ds + \int_{t_0}^{t} F'(u(x,s))u_x(x,s)\,ds \;\in C^0((0,T]\times\overline{\Omega}).$$

Hence,

$$u(x,t) - u(x,t_0) - \int_{t_0}^{t} u_{xx}(x,s)\,ds + \int_{t_0}^{t} F'(u(x,s))u_x(x,s)\,ds = 0$$

holds classically as functions in $C^0((0,T]\times\overline{\Omega})$. In particular, we have

$$u_t - u_{xx} - F'(u)u_x = 0$$

holds as a classical differential equation in the space $L^2(\Omega)$.

Let $\widetilde{u}(x,t) = u(x,t) - \max_{0\le x\le 1} u(x,t_0)$ and $\widetilde{u}^+(x,t) = \max\{0, \widetilde{u}(x,t)\}$. It is clear from the above that $\widetilde{u}(x,t)$ satisfies:

$$\widetilde{u}_t - \widetilde{u}_{xx} + F'(u)\widetilde{u}_x = 0, \tag{8}$$

$$\widetilde{u}_x(0,t) = \widetilde{u}_x(1,t) = 0, \tag{9}$$

By taking the L^2 inner product of equation (8) with $\widetilde{u}^+$, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\widetilde{u}^+)^2 dx + \int_0^1 ((\widetilde{u}^+)_x)^2 dx + \int_0^1 F'(u)\widetilde{u}^+(\widetilde{u}^+)_x\, dx = 0.$$

Since $u \in C((0,T]\times\overline{\Omega})$, we obtain

$$\left|\int_0^1 F'(u)\widetilde{u}^+(\widetilde{u}^+)_x\, dx\right| \le \max_{0\le x\le 1} |F'(u(x,t))|\; \|\widetilde{u}^+\|_{L^2(\Omega)}\; \|(\widetilde{u}^+)_x\|_{L^2(\Omega)}.$$

By Cauchy–Schwarz inequality, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\widetilde{u}^+)^2 dx \le \max_{0\le x\le 1} |F'(u(x,t))|^2 \int_0^1 (\widetilde{u}^+)^2 dx.$$

Thanks to Gronwall's inequality,

$$\|\widetilde{u}^+(t)\|^2_{L^2(\Omega)} \le \|\widetilde{u}^+(t_0)\|^2_{L^2(\Omega)}\, e^{\int_{t_0}^{t} \max_{0\le x\le 1} |F'(u(x,\tau))|^2 \; d\tau}.$$

since $\widetilde{u}^+ = 0$, we have $\|\widetilde{u}^+(t)\|_{L^2(\Omega)} = 0$ for all $t \ge t_0 > 0$. In other words,

$$\max_{0\le x\le 1} u(x,t) \le \max_{0\le x\le 1} u(x,t_0), \qquad \text{for all } t \ge t_0 > 0.$$

Similarly, we can show (7), for all $t \ge t_0 > 0$.

Next, we show that (6) and (7) hold for $t_0 = 0$. Let us denote by

$$M(t) = \max_{0\le x\le 1} u(x,t) \qquad \text{and } m(t) = \min_{0\le x\le 1} u(x,t).$$

Since (6) and (7) hold for $t_0 > 0$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} = \max\{|M(t)|, |m(t)|\}$$

is a monotonic nonincreasing function of time for $t > 0$. Therefore,

$$\lim_{t\downarrow 0} \|u(\cdot,t)\|_{L^\infty(\Omega)} = M^0_\infty < \infty. \tag{10}$$

Moreover, since we proved that (6) holds for $t_0 > 0$, then $M(t)$ is a monotonic nonincreasing function of time for $t > 0$. As a result of (10) we have

$$\lim_{t \downarrow 0} M(t) = M^0 < \infty.$$

From the definition of regular solution $u \in C([0,T], L^2)$, therefore,

$$\lim_{t \downarrow 0} \|u(\cdot, t) - u_{\text{in}}(\cdot)\|_{L^2(\Omega)} = 0.$$

Thus,

$$\lim_{t \downarrow 0} \| \big(u(\cdot, t) - M^0\big)^+ - \big(u_{\text{in}}(\cdot) - M^0\big)^+ \|_{L^2(\Omega)} = 0. \tag{11}$$

Observe, however, that

$$u(x,t) - M^0 \leq M(t) - M^0$$

and since $M(t) - M^0 \geq 0$ we conclude that

$$0 \leq \big(u(\cdot, t) - M^0\big)^+ \leq M(t) - M^0.$$

Thus,

$$\lim_{t \downarrow 0} \| \big(u(\cdot, t) - M^0\big)^+ \|_{L^2(\Omega)} = 0.$$

Together with (11) implies that

$$\big(u_{\text{in}}(\cdot) - M^0\big)^+ = 0,$$

and hence

$$M(0) = \max_{0 \leq x \leq 1} u_{\text{in}}(x) \leq M^0 \leq M(t) \qquad \text{for all } t > 0,$$

which proves (6) for all $t_0 \geq 0$.

Once again, the proof of (7) is similar. □

Theorem 3. *Let T be any positive number. Then, for every $u_{\text{in}}(x) \in C^0(\overline{\Omega})$, there is a unique regular solution $u(x,t)$.*

We will give the proof of Theorem 3 later. But, first, let us consider the following initial boundary value problem (linearization of (1)–(3) about $\eta(x,t)$)

$$\frac{\partial \xi}{\partial t} - \frac{\partial^2 \xi}{\partial x^2} + F'(\eta(x,t)) \frac{\partial \xi}{\partial x} = 0, \qquad x \in \Omega,\ t > 0, \tag{12}$$

$$\xi_x(0,t) = \xi_x(1,t) = 0, \tag{13}$$

$$\xi(x,0) = u_{\text{in}}(x), \qquad x \in \Omega, \tag{14}$$

where $\eta(x,t)$ is a given function such that

$$\widetilde{\eta} = \sup_{\substack{0 \leq x \leq 1 \\ t \geq 0}} |\eta(x,t)| < \infty \tag{15}$$

and, for every $T > 0$,

$$\widetilde{\eta}_T = \left(\int_0^T \|\eta(\cdot, t)\|_{H^1}^2 \, dt \right)^{1/2} < \infty. \tag{16}$$

Definition 4. *Let $u_{\text{in}} \in C^0(\overline{\Omega})$ and let T be any positive number. A regular solution of* (12)–(14) *on* $[0,T]$ *is a function $\xi(x,t)$ such that:*

$$\xi \in L^2_{\text{loc}}((0,T];H^3(\Omega)) \cap C((0,T];H^2(\Omega)) \cap L^2([0,T];H^1(\Omega)) \cap C([0,T];L^2(\Omega))$$

and

$$\langle \xi(t),\phi\rangle_{L^2} - \langle \xi(t_0),\phi\rangle_{L^2} + \int_{t_0}^{t} \langle \xi_x(s),\phi_x\rangle_{L^2}\,ds + \tag{17}$$
$$+ \int_{t_0}^{t} \langle F'(\eta(s))\xi_x(s),\phi\rangle_{L^2}\,ds = 0,$$

for every $\phi \in \mathcal{V}$ and every $t,\ t_0 \in [0,T]$.

Proposition 5. *Assume that η satisfies* (15) *and* (16)*, and $u_{\text{in}}(x) \in C^0(\overline{\Omega})$. Then for any $T>0$ there is a unique regular solution $\xi(x,t)$ of the system* (12)–(14) *such that for every $t \in (0,T]$*

$$\|\xi(\cdot,t)\|^2_{L^2} + \int_0^t \|\xi_x(\cdot,s)\|^2_{L^2}\,ds \le K_1(T,\widetilde{\eta}) \tag{18}$$

$$\|\xi_x(\cdot,t)\|^2_{L^2} + \frac{1}{t}\int_0^t \|\xi_{xx}(\cdot,s)\|^2_{L^2} s\,ds \le K_2(T,\widetilde{\eta}) \tag{19}$$

$$\|\xi_{xx}(\cdot,t)\|^2_{L^2} + \frac{1}{t^2}\int_0^t \|\xi_{xxx}(\cdot,s)\|^2_{L^2} s^2\,ds \le K_3(T,\widetilde{\eta},\widetilde{\eta}_T), \tag{20}$$

where $K_1(T,\widetilde{\eta}), K_2(T,\widetilde{\eta})$ and $K_3(T,\widetilde{\eta},\widetilde{\eta}_T)$ will be specified later as in (26), (28) *and* (30)*, respectively. Furthermore, $\xi(x,t)$ satisfies the maximum principle:*

$$\|\xi(\cdot,t)\|_{L^\infty} \le \|u_{\text{in}}\|_{L^\infty} = M_{\text{in}} \qquad \textit{for all} \quad t \ge 0. \tag{21}$$

Proof. We prove this Proposition by using the standard Galerkin procedure. It is clear that $\phi_0 = 1, \phi_k = \sqrt{\frac{2}{\pi}}\cos k\pi x, k = 1,2,\cdots$, form an orthonormal basis of $L^2(\Omega)$. Let H_m be the space of functions spanned by $\{\phi_k\}_{k=0}^m$, and let $P_m : L^2(\Omega) \to H_m$ be the L^2 orthogonal projection onto H_m. The Galerkin approximating equation associated with system (12)–(14) reads:

$$\frac{\partial \xi_m}{\partial t} - \frac{\partial^2 \xi_m}{\partial x^2} + P_m\left[F'(\eta(x,t))\frac{\partial \xi_m}{\partial x}\right] = 0, \tag{22}$$

$$(\xi_m)_x(0,t) = (\xi_m)_x(1,t) = 0, \tag{23}$$

$$\xi_m(x,0) = P_m u_{\text{in}}(x), \tag{24}$$

where $\xi_m(x,t) = \sum_{k=0}^m a_k(t)\phi_k(x) \in H_m$. The global existence of solution to this finite dimensional linear system (22)–(24) is obvious. We only show the regularity, i.e. the estimates (18)–(20).

Let $T>0$ be given. By taking the L^2 inner product of equation (22) with ξ_m, we get

$$\frac{1}{2}\frac{d\|\xi_m\|^2_{L^2}}{dt} + \|(\xi_m)_x\|^2_{L^2} = -\int_0^1 F'(\eta(x,t))(\xi_m)_x\,\xi_m\,dx \le \widetilde{F}(\widetilde{\eta})\|\xi_m\|_{L^2}\|(\xi_m)_x\|_{L^2},$$

where $\widetilde{F}$ and $\widetilde{\eta}$ are as in (4) and (15), respectively. By using the Young's inequality we obtain

$$\frac{d\|\xi_m\|_{L^2}^2}{dt} + \|(\xi_m)_x\|_{L^2}^2 \le (\widetilde{F}(\widetilde{\eta}))^2\|\xi_m\|_{L^2}^2.$$

Thanks to Gronwall's inequality we have for every $t \in [0,T]$,

$$\|\xi_m(\cdot,t)\|_{L^2}^2 + \int_0^t \|(\xi_m)_x(x,s)\|_{L^2}^2\, ds \le K_1(T,\widetilde{\eta}), \tag{25}$$

where

$$K_1(T,\widetilde{\eta}) = \|u_{\text{in}}\|_{L^2}^2 e^{(\widetilde{F}(\widetilde{\eta}))^2 T}. \tag{26}$$

By taking the L^2 inner product of equation (22) with $-(\xi_m)_{xx}$, we reach

$$\begin{aligned}&\frac{1}{2}\frac{d\|(\xi_m)_x\|_{L^2}^2}{dt} + \int_0^1 |(\xi_m)_{xx}|^2\, dx = \int_0^1 F'(\eta(x,t))(\xi_m)_x(\xi_m)_{xx}\, dx\\ &\le \widetilde{F}(\widetilde{\eta})\|(\xi_m)_x\|_{L^2}\|(\xi_m)_{xx}\|_{L^2},\end{aligned}$$

where, again, $\widetilde{F}$ and $\widetilde{\eta}$ are as in (4) and (15), respectively. Again, by using Young's inequality we obtain

$$\frac{d\|(\xi_m)_x\|_{L^2}^2}{dt} + \|(\xi_m)_{xx}\|_{L^2}^2 \le (\widetilde{F}(\widetilde{\eta}))^2\|(\xi_m)_x\|_{L^2}^2.$$

Thanks to Gronwall's inequality we get for $0 < \tau \le t \le T$,

$$\|(\xi_m)_x(\cdot,t)\|_{L^2}^2 + \int_\tau^t \|(\xi_m)_{xx}(\cdot,s)\|_{L^2}^2 \le e^{(\widetilde{F}(\widetilde{\eta}))^2 T}\|(\xi_m)_x(\cdot,\tau)\|_{L^2}^2.$$

By integrating the above inequality, with respect to τ over the interval $[0,t]$, and applying estimate (25), we get

$$\|(\xi_m)_x(\cdot,t)\|_{L^2}^2 + \frac{1}{t}\int_0^t \|(\xi_m)_{xx}(\cdot,s)\|_{L^2}^2\, s\, ds \le K_2(T,\widetilde{\eta}), \tag{27}$$

where

$$K_2(T,\widetilde{\eta}) = \frac{K_1(T,\widetilde{\eta})}{t} e^{(\widetilde{F}(\widetilde{\eta}))^2 T}, \tag{28}$$

and $K_1(T,\widetilde{\eta})$ is as in (26). Once more, by taking the L^2 inner product of equation (22) with $(\xi_m)_{xxxx}$, we reach

$$\frac{1}{2}\frac{d\|(\xi_m)_{xx}\|_{L^2}^2}{dt} - \int_0^1 (\xi_m)_{xx}(\xi_m)_{xxxx}\, dx + \int_0^1 F'(\eta(x,t))\,(\xi_m)_x\,(\xi_m)_{xxxx}\, dx = 0.$$

By integrating by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d\|(\xi_m)_{xx}\|_{L^2}^2}{dt} + \|(\xi_m)_{xxx}\|_{L^2}^2\\ &= \int_0^1 F''(\eta(x,t))\eta_x(x,t)(\xi_m)_x(\xi_m)_{xxx}\, dx + \int_0^1 F'(\eta(x,t))(\xi_m)_{xx}(\xi_m)_{xxx}\, dx\\ &\le \widetilde{F}(\widetilde{\eta})\left[\,\|\eta_x\|_{L^2}\|(\xi_m)_x\|_{L^\infty} + \|(\xi_m)_{xx}\|_{L^2}\,\right]\|(\xi_m)_{xxx}\|_{L^2}.\end{aligned}$$

By using Young's inequality, we get

$$\frac{d\|(\xi_m)_{xx}\|_{L^2}^2}{dt} + \|(\xi_m)_{xxx}\|_{L^2}^2 \le 2(\widetilde{F}(\widetilde{\eta}))^2\left[\,\|\eta_x\|_{L^2}^2\|(\xi_m)_x\|_{L^\infty}^2 + \|(\xi_m)_{xx}\|_{L^2}^2\,\right].$$

Notice that $(\xi_m)_x(0,t)=0$. Then

$$\|(\xi_m)_x(x,t)\|_{L^\infty} = \max_{0\le x\le 1}\left|\int_0^x (\xi_m)_{xx}(y,t)\,dy\right| \le \int_0^1 |(\xi_m)_{xx}(y,t)|\,dy \le \|(\xi_m)_{xx}\|_{L^2}.$$

Therefore,

$$\frac{d\|(\xi_m)_{xx}\|_{L^2}^2}{dt} + \|(\xi_m)_{xxx}\|_{L^2}^2 \le 2(\widetilde{F}(\widetilde{\eta}))^2\left[\,\|\eta_x\|_{L^2}^2+1\,\right]\|(\xi_m)_{xx}\|_{L^2}^2.$$

Thanks to Gronwall's inequality, for $0<\tau\le t\le T$,

$$\begin{aligned}
&\|(\xi_m)_{xx}(\cdot,t)\|_{L^2}^2 + \int_\tau^t \|(\xi_m)_{xxx}(\cdot,s)\|_{L^2}^2\,ds \\
&\le \|(\xi_m)_{xx}(\cdot,\tau)\|_{L^2}^2\, e^{2(\widetilde{F}(\widetilde{\eta}))^2\int_0^T\left[\,\|\eta_x(\cdot,\tau)\|_{L^2}^2+1\,\right]d\tau} \\
&\le \|(\xi_m)_{xx}(\cdot,\tau)\|_{L^2}^2\, e^{2(\widetilde{F}(\widetilde{\eta}))^2\left[\,\widetilde{\eta}_T+T\,\right]}
\end{aligned}$$

where $\widetilde{\eta}_T$ is as in (16). By multiplying the above inequality by τ, and integrating with respect to τ over $[0,t]$, and applying estimate (27), we obtain

$$\|(\xi_m)_{xx}(\cdot,t)\|_{L^2}^2 + \frac{1}{t^2}\int_0^t \|(\xi_m)_{xxx}(\cdot,s)\|_{L^2}^2\,s^2\,ds \le K_3(T,\widetilde{\eta},\widetilde{\eta}_T), \tag{29}$$

where

$$K_3(T,\widetilde{\eta},\widetilde{\eta}_T) = \frac{2K_2(T,\widetilde{\eta})}{t}e^{2(\widetilde{F}(\widetilde{\eta}))^2\left[\,\widetilde{\eta}_T+T\,\right]}, \tag{30}$$

and $K_2(T,\widetilde{\eta})$ is as in (28).

Finally, by using the appropriate "Compactness Theorems" one can extract a subsequence $\{\xi_{m'}(x,t)\}$ which converges to the limit function $\xi(x,t)$ that solves the system (12)–(14), satisfies (17), and belongs to

$$\xi \in L^2_{\text{loc}}((0,T];H^3(\Omega)) \cap C((0,T];H^2(\Omega)) \cap L^2([0,T];H^1(\Omega)) \cap C([0,T];L^2(\Omega)).$$

Furthermore, from (25), (27) and (29) we conclude that the limit function ξ satisfies (18)–(20). To prove (21) (i.e. the maximum principle) one can follow, almost step by step, the proof of Proposition 2.

Next, we show the uniqueness of ξ. Suppose that $\overline{\xi}$ and $\overline{\overline{\xi}}$ are two regular solutions of system (12)–(14). Let $\chi=\overline{\overline{\xi}}-\overline{\xi}$. Since the system (12)–(14) is a linear system, we have that χ is a regular solution of the system (12)–(14) but with $u_{\text{in}}(x)=0$. It is clear that χ satisfies (18). However, in the case of $u_{\text{in}}(x)=0$, by applying (26), we get $K_1(T,\widetilde{\eta})=0$. Therefore,

$$\|\chi(\cdot,t)\|_{L^2}=0,$$

and $\chi=\overline{\overline{\xi}}-\overline{\xi}=0$. Therefore, the regular solution is unique. □

Now, we are ready to prove Theorem 3.

Proof. Let $T > 0$ be given. First, let us consider, for $n \geq 1$, the following systems

$$\frac{\partial u^{(n)}}{\partial t} - \frac{\partial^2 u^{(n)}}{\partial x^2} + F'(u^{(n-1)})\frac{\partial u^{(n)}}{\partial x} = 0, \tag{31}$$
$$u_x^{(n)}(0,t) = u_x^{(n)}(1,t) = 0, \tag{32}$$
$$u^{(n)}(x,0) = u_{\text{in}}(x), \tag{33}$$
$$u^{(0)} = 0. \tag{34}$$

Denote by $\{u^{(n)}(x,t)\}_{n=0}^{\infty}$, the sequence of functions, that solve the above systems (31)–(34) for $n = 1, 2, \cdots$, respectively. By applying Proposition 5 in the case of $\eta = 0$, we get

$$u^{(1)} \in L^2_{\text{loc}}((0,T]; H^3(\Omega)) \cap C((0,T]; H^2(\Omega)) \cap L^2([0,T]; H^1(\Omega)) \cap C([0,T]; L^2(\Omega)),$$

and the estimates (18)–(21) hold for $\xi = u^{(1)}$. Therefore, by induction, we apply Proposition 5 in the case of $\eta = u^{(n-1)}$ and we are able to show that

$$u^{(n)} \in L^2_{\text{loc}}((0,T]; H^3(\Omega)) \cap C((0,T]; H^2(\Omega)) \cap L^2([0,T]; H^1(\Omega)) \cap C([0,T]; L^2(\Omega)),$$

with $u^{(n)}$ satisfies *the maximum principle*, i.e.,

$$\|u^{(n)}(\cdot,t)\|_{L^\infty} \leq \|u_{\text{in}}\|_{L^\infty} = M_{\text{in}}, \tag{35}$$

and

$$\|u^{(n)}(\cdot,t)\|_{L^2}^2 + \int_0^t \|u_x^{(n)}(\cdot,s)\|_{L^2}^2\, ds \leq K_1(T, M_{\text{in}}) \tag{36}$$
$$\|u_x^{(n)}(\cdot,t)\|_{L^2}^2 + \frac{1}{t}\int_0^t \|u_{xx}^{(n)}(\cdot,s)\|_{L^2}^2 s ds \leq K_2(T, M_{\text{in}}) \tag{37}$$
$$\|u_{xx}^{(n)}(\cdot,t)\|_{L^2}^2 + \frac{1}{t^2}\int_0^t \|u_{xxx}^{(n)}(\cdot,s)\|_{L^2}^2 s^2 ds \leq K_3(T, M_{\text{in}}, K_1(T, M_{\text{in}})), \tag{38}$$

where M_{in} is as in (35), and $K_1(T, M_{\text{in}}), K_2(T, M_{\text{in}})$ and $K_3(T, M_{\text{in}}, K_1(T, M_{\text{in}}))$ are as in (26), (28) and (30), respectively. Next, we show that $\{u^{(n)}(x,t)\}_{n=0}^{\infty}$ is a Cauchy sequence in $L^2((0,T; H^1(\Omega)) \cap C([0,T]; H)$. Denote by

$$\widetilde{u}^{(n)} = u^{(n)} - u^{(n-1)}, \qquad \text{for } n = 2, 3, \cdots.$$

It is easy to get

$$\frac{\partial \widetilde{u}^{(n)}}{\partial t} - \frac{\partial^2 \widetilde{u}^{(n)}}{\partial x^2} + F'(u^{(n-1)})\frac{\partial \widetilde{u}^{(n)}}{\partial x} + \left[F'(u^{(n-1)}) - F'(u^{(n-2)})\right]\frac{\partial u^{(n-1)}}{\partial x} = 0 \tag{39}$$
$$\widetilde{u}_x^{(n)}(0,t) = \widetilde{u}_x^{(n)}(1,t) = 0 \tag{40}$$
$$\widetilde{u}^{(n)}(x,0) = 0. \tag{41}$$

By taking the L^2 inner product of equation (39) with $\widetilde{u}^{(n)}$, we get

$$\begin{aligned}
\frac{1}{2}\frac{d\|\widetilde{u}^{(n)}\|_{L^2}^2}{dt} + \left\|(\widetilde{u}^{(n)})_x\right\|_{L^2}^2 &= -\int_0^1 F'(u^{(n-1)})(\widetilde{u}^{(n)})_x\, \widetilde{u}^{(n)}\, dx - \\
&- \int_0^1 \left[F'(u^{(n-1)}) - F'(u^{(n-2)})\right](u^{(n-1)})_x \widetilde{u}^{(n)}\, dx \\
&\leq \widetilde{F}(M_{\text{in}})\|\widetilde{u}^{(n)}\|_{L^2}\|(\widetilde{u}^{(n)})_x\|_{L^2} + \widetilde{F}(M_{\text{in}})\|\widetilde{u}^{(n-1)}\|_{L^2}\|(u^{(n-1)})_x\|_{L^\infty}\|\widetilde{u}^{(n)}\|_{L^2},
\end{aligned}$$

where $\widetilde{F}$ and M_{in} are as in (4) and (35), respectively. Recall that

$$(u^{(n-1)})_x(0,t) = (u^{(n-1)})_x(1,t) = 0.$$

Then, by applying the following version Sobolev inequality

$$\|f_x\|_{L^\infty} \le \|f_x\|_{L^2}^{1/2} \|f_{xx}\|_{L^2}^{1/2}, \qquad \forall f \in \mathcal{V}, \tag{42}$$

and by using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} &\frac{d\|\widetilde{u}^{(n)}\|_{L^2}^2}{dt} + \left\|(\widetilde{u}^{(n)})_x\right\|_{L^2}^2 \le (\widetilde{F}(M_{\text{in}}))^2 \|\widetilde{u}^{(n)}\|_{L^2}^2 + \\ &\quad + 2\widetilde{F}(M_{\text{in}}) \|\widetilde{u}^{(n-1)}\|_{L^2} \|(u^{(n-1)})_x\|_{L^2}^{1/2} \|(u^{(n-1)})_{xx}\|_{L^2}^{1/2} \|\widetilde{u}^{(n)}\|_{L^2} \\ &\le (\widetilde{F}(M_{\text{in}}))^2 \left[1 + \|(u^{(n-1)})_{xx}\|_{L^2}\right] \|\widetilde{u}^{(n)}\|_{L^2}^2 + \|\widetilde{u}^{(n-1)}\|_{L^2}^2 \|(u^{(n-1)})_x\|_{L^2}. \end{aligned}$$

Thanks to Gronwall's inequality, for every $0 \le t_0 \le t \le T$,

$$\begin{aligned} \|\widetilde{u}^{(n)}(\cdot,t)\|_{L^2}^2 + \int_{t_0}^t \left\|(\widetilde{u}^{(n)})_x(\cdot,s)\right\|_{L^2}^2 ds \le e^{\int_{t_0}^t \Psi(\tau)\, d\tau} \|\widetilde{u}^{(n)}(\cdot,t_0)\|_{L^2}^2 + \\ + \int_{t_0}^t e^{\int_s^t \Psi(\tau)\, d\tau} \|\widetilde{u}^{(n-1)}(\cdot,s)\|_{L^2}^2 \|(u^{(n-1)})_x(\cdot,s)\|_{L^2}\, ds \end{aligned} \tag{43}$$

where

$$\Psi(\tau) = (\widetilde{F}(M_{\text{in}}))^2 \left[1 + \|(u^{(n-1)})_{xx}(\cdot,\tau)\|_{L^2}\right]. \tag{44}$$

On the other hand,

$$\begin{aligned} &\int_s^t \|(u^{(n-1)})_{xx}(\cdot,\tau)\|_{L^2}\, d\tau \le \left(\int_s^t \tau\, \|(u^{(n-1)})_{xx}(\cdot,\tau)\|_{L^2}^2\, d\tau\right)^{\frac12} \left(\int_s^t \frac{1}{\tau}\, d\tau\right)^{\frac12} \\ &\quad \le \left(K_1(T, M_{\text{in}}) e^{(\widetilde{F}(M_{\text{in}}))^2 T}\right)^{\frac12} \left(\ln \frac{t}{s}\right)^{\frac12}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{t_0}^t e^{(\widetilde{F}(M_{\text{in}}))^2 \int_s^t \|(u^{(n-1)})_{xx}(\cdot,\tau)\|_{L^2}\, d\tau} \|(u^{(n-1)})_x(\cdot,s)\|_{L^2}\, ds \\ &\le \int_{t_0}^t e^{(\widetilde{F}(M_{\text{in}}))^2 \left(K_1(T, M_{\text{in}}) e^{(\widetilde{F}(M_{\text{in}}))^2 T}\right)^{\frac12} \left(\ln \frac{t}{s}\right)^{\frac12}} \|(u^{(n-1)})_x(\cdot,s)\|_{L^2}\, ds \\ &\le \left(\int_{t_0}^t \|(u^{(n-1)})_x(\cdot,s)\|_{L^2}^2\, ds\right)^{\frac12} \times \\ &\quad \times \left(\int_{t_0}^t e^{2(\widetilde{F}(M_{\text{in}}))^2 \left(K_1(T, M_{\text{in}}) e^{(\widetilde{F}(M_{\text{in}}))^2 T}\right)^{\frac12} \left(\ln \frac{t}{s}\right)^{\frac12}} ds\right)^{\frac12}. \end{aligned} \tag{45}$$

In the case of $t_0 > 0$, thanks to (36), we have

$$\begin{aligned}
&\int_{t_0}^{t} e^{(\widetilde{F}(M_{\text{in}}))^2 \int_s^t \|(u^{(n-1)})_{xx}(\cdot,\tau)\|_{L^2}\, d\tau} \|(u^{(n-1)})_x(\cdot,s)\|_{L^2}\, ds \\
&\le \left(K_1(T,M_{\text{in}})\right)^{\frac{1}{2}} \times \\
&\qquad \times \left(e^{2(\widetilde{F}(M_{\text{in}}))^2 \left(K_1(T,M_{\text{in}}) e^{(\widetilde{F}(M_{\text{in}}))^2 T}\right)^{\frac{1}{2}} \left(\ln \frac{T}{t_0}\right)^{\frac{1}{2}}} (t-t_0) \right)^{\frac{1}{2}} \\
&\le \left(K_1(T,M_{\text{in}}) K_4(T,M_{\text{in}})\right)^{\frac{1}{2}} \ (t-t_0)^{\frac{1}{2}},
\end{aligned} \tag{46}$$

where

$$K_4(T,M_{\text{in}}) = e^{2(\widetilde{F}(M_{\text{in}}))^2 \left(K_1(T,M_{\text{in}}) e^{(\widetilde{F}(M_{\text{in}}))^2 T}\right)^{\frac{1}{2}} \left(\ln \frac{T}{t_0}\right)^{\frac{1}{2}}} \tag{47}$$

In the case of $t_0 = 0$, thanks to (36) and (45), we reach

$$\begin{aligned}
&\int_{t_0}^{t} e^{(\widetilde{F}(M_{\text{in}}))^2 \int_s^t \|(u^{(n-1)})_{xx}(\cdot,\tau)\|_{L^2}\, d\tau} \|(u^{(n-1)})_x(\cdot,s)\|_{L^2}\, ds \\
&\le \left(K_1(T,M_{\text{in}})\right)^{\frac{1}{2}} \times \\
&\quad \times \left(\sum_{j=1}^{\infty} \int_{\frac{t}{j+1}}^{\frac{t}{j}} e^{2(\widetilde{F}(M_{\text{in}}))^2 \left(K_1(T,M_{\text{in}}) e^{(\widetilde{F}(M_{\text{in}}))^2 T}\right)^{\frac{1}{2}} \left(\ln \frac{t}{s}\right)^{\frac{1}{2}}} ds \right)^{\frac{1}{2}} \\
&\le \left(K_1(T,M_{\text{in}})\right)^{\frac{1}{2}} \times \\
&\quad \times \left(\sum_{j=1}^{\infty} e^{2(\widetilde{F}(M_{\text{in}}))^2 \left(K_1(T,M_{\text{in}}) e^{(\widetilde{F}(M_{\text{in}}))^2 T}\right)^{\frac{1}{2}} (\ln(j+1))^{\frac{1}{2}}} \left(\frac{t}{j} - \frac{t}{j+1}\right) \right)^{\frac{1}{2}} \\
&\le \left(K_1(T,M_{\text{in}}) K_5(T,M_{\text{in}})\right)^{\frac{1}{2}} \ t^{\frac{1}{2}},
\end{aligned}$$

where

$$\begin{aligned}
&K_5(T,M_{\text{in}}) = \\
&= \sum_{j=1}^{\infty} \frac{e^{2(\widetilde{F}(M_{\text{in}}))^2 \left(K_1(T,M_{\text{in}}) e^{(\widetilde{F}(M_{\text{in}}))^2 T}\right)^{\frac{1}{2}} (\ln(j+1))^{1/2}}}{j(j+1)} < \infty.
\end{aligned} \tag{48}$$

Therefore, when $t_0 = 0$, we get from the above and (36)

$$\begin{aligned}
&\|\widetilde{u}^{(n)}(\cdot,t)\|_{L^2}^2 + \int_0^t \left\|(\widetilde{u}^{(n)})_x(\cdot,s)\right\|_{L^2}^2 ds \\
&\le e^{(\widetilde{F}(M_{\text{in}}))^2 T} \left(K_1(T,M_{\text{in}}) K_5(T,M_{\text{in}})\right)^{\frac{1}{2}} \ t^{\frac{1}{2}} \max_{0\le s\le t} \|\widetilde{u}^{(n-1)}(\cdot,s)\|_{L^2}^2,
\end{aligned}$$

and when $t_0 > 0$, we obtain

$$\|\widetilde{u}^{(n)}(\cdot,t)\|_{L^2}^2 + \int_{t_0}^t \left\|(\widetilde{u}^{(n)})_x(\cdot,s)\right\|_{L^2}^2 ds$$

$$\leq \|\widetilde{u}^{(n)}(\cdot,t_0)\|_{L^2}^2 \, e^{(\widetilde{F}(M_{\text{in}}))^2\left[T + \left(K_1(T,M_{\text{in}})e^{(\widetilde{F}(M_{\text{in}}))^2 T}\right)^{\frac{1}{2}} \left(\ln \frac{T}{t_0}\right)^{\frac{1}{2}}\right]} +$$

$$+ e^{(\widetilde{F}(M_{\text{in}}))^2 T} \left(K_1(T,M_{\text{in}})K_4(T,M_{\text{in}})\right)^{\frac{1}{2}} (t-t_0)^{\frac{1}{2}} \max_{0\leq s\leq t} \|\widetilde{u}^{(n-1)}(\cdot,s)\|_{L^2}^2. \quad (49)$$

Let

$$T^* = \frac{1}{4e^{2(\widetilde{F}(M_{\text{in}}))^2 T} K_1(T,M_{\text{in}})[K_4(T,M_{\text{in}}) + K_5(T,M_{\text{in}})]}. \quad (50)$$

Then, from the above we have

$$\max_{0\leq t\leq T^*} \|\widetilde{u}^{(n)}(\cdot,t)\|_{L^2}^2 + \int_0^{T^*} \left\|(\widetilde{u}^{(n)})_x(\cdot,t)\right\|_{L^2}^2 dt$$

$$\leq \frac{1}{2} \max_{0\leq t\leq T^*} \|\widetilde{u}^{(n-1)}(\cdot,t)\|_{L^2}^2.$$

By induction, we get

$$\max_{0\leq t\leq T^*} \|\widetilde{u}^{(n)}(\cdot,t)\|_{L^2}^2 + \int_0^{T^*} \left\|(\widetilde{u}^{(n)})_x(\cdot,t)\right\|_{L^2}^2 dt \leq \frac{1}{2^{n-2}} M_{\text{in}}^2.$$

Thus, $\{u^{(n)}(x,t)\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(0,T^*;H^1(\Omega)) \cap C([0,T^*];L^2(\Omega))$. By taking $t_0 = T^*$ in equation (49) and by noticing that $\{u^{(n)}(x,T^*)\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(\Omega)$, one can easily see that $\{u^{(n)}(x,t)\}_{n=0}^\infty$ is also a Cauchy sequence in $L^2(T^*,2T^*;H^1(\Omega)) \cap C([T^*,2T^*];L^2(\Omega))$. We repeat the above procedure again and again until we cover the interval $[0,T]$ with finitely many steps, which implies that $\{u^{(n)}(x,t)\}_{n=0}^\infty$ is a Cauchy sequence in $L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$. I.e.,

$$u(x,t) = \lim_{n\to\infty} u^{(n)}(x,t),$$

in $L^2(0,T;H^1(\Omega)) \cap C([0,T];H)$. Moreover, $u(x,t)$ satisfies the weak formulation (17) and

$$\max_{\substack{0\leq x\leq 1\\ t\geq 0}} |u(x,t)| \leq M_{\text{in}} \quad (51)$$

$$\int_0^T \|u(x,t)\|_{H^1}^2 dt \leq K_1(T,M_{\text{in}}). \quad (52)$$

Therefore, $u(x,t)$ satisfies the condition (15)–(16). By applying the Proposition 5, we have

$$u \in L^2_{\text{loc}}((0,T];H^3(\Omega)) \cap C((0,T];H^2(\Omega)) \cap L^2([0,T];H^1(\Omega)) \cap C([0,T];L^2(\Omega)).$$

Next we show the uniqueness. Suppose that u_1 and u_2 are two regular solutions. Let $u = u_2 - u_1$, which satisfies

$$u_t - u_{xx} + F'(u_2)u_x + (F'(u_2) - F'(u_1))(u_1)_x = 0 \quad (53)$$

$$u_x(0,t) = u_x(1,t) = 0, \quad (54)$$

$$u(x,0) = 0. \quad (55)$$

By taking the L^2 inner product of equation (53) with u, we get

$$\frac{1}{2}\frac{d\|u\|_{L^2}^2}{dt} + \|u_x\|_{L^2}^2 = -\int_0^1 F'(u_2)u_x u\,dx - \int_0^1 \left[F'(u_2) - F'(u_1)\right](u_1)_x u\,dx$$
$$\le \widetilde{F}(M_{\text{in}})\|u\|_{L^2}\|u_x\|_{L^2} + \widetilde{F}(M_{\text{in}})\|(u_1)_x\|_{L^2}\|u\|_{L^\infty}\|u\|_{L^2}.$$

Notice that

$$|u(x,t)| \le \left|\int_0^1 u(y,t)\,dy\right| + \|u_x\|_{L^2} \le \|u\|_{L^2} + \|u_x\|_{L^2}.$$

therefore,

$$\frac{1}{2}\frac{d\|u\|_{L^2}^2}{dt} + \|u_x\|_{L^2}^2 \le \widetilde{F}(M_{\text{in}})[(1 + \|(u_1)_x\|_{L^2})\|u\|_{L^2}\|u_x\|_{L^2} + \|(u_1)_x\|_{L^2}\|u\|_{L^2}^2].$$

By using the Cauchy–Schwarz inequality, we obtain

$$\frac{d\|u\|_{L^2}^2}{dt} \le \frac{(\widetilde{F}(M_{\text{in}}))^2}{2}[1 + \|(u_1)_x\|_{L^2}]^2\|u\|_{L^2}^2 + 2\widetilde{F}(M_{\text{in}})\|(u_1)_x\|_{L^2}\|u\|_{L^2}^2.$$

Thanks to Gronwall's inequality we have

$$\|u\|_{L^2}^2 = 0.$$

Therefore, the regular solution is unique.

□

Remark In the case of the viscous Burgers equation, the authors of [16] take advantage of the special form of $F = \frac{u^2}{2}$ to show similar global existence result for initial data $u_{\text{in}} \in L^2(\Omega)$.

3. Long–time Dynamics

In the previous section we have established the global existence and uniqueness of regular solution for (1)–(3). In this section we study the long–time behavior of this solution. But first let us identify the set of steady state solutions for (1)–(2).

Proposition 6. *$u(x)$ is a steady state solution of (1)–(2) if and only if $u(x) \equiv$ const.*

Proof. First, notice that every constant function $u(x) \equiv$ const is a steady state solution. Next we show that all steady state solutions are constant functions. Let $u(x)$ be any steady state of equations (1)−(2). Then $u(x)$ satisfies

$$-u_{xx} + (F(u))_x = 0 \tag{56}$$
$$u_x(0) = u_x(1) = 0. \tag{57}$$

By integrating (56), we get

$$u_x(x) = u_x(0)e^{\int_0^x F'(u(s))ds}.$$

Since $u_x(0) = 0$, then $u_x \equiv 0$. Therefore, $u(x)$ is a constant function. □

Since we are interested in the asymptotic behavior of solutions of the system (1)–(3), we will study the system (1)–(3) when t is large enough. From now on we assume that $t \ge t_0 > 0$, for t_0 large enough, as large as needed. Recall that for almost every $t \ge t_0 > 0, u(\cdot,t) \in H^3(\Omega)$.

Denote by $v = u_x$. Then v satisfies

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} + F''(u)v^2 + F'(u)\frac{\partial v}{\partial x} = 0 \qquad \forall x \in \Omega,\ t > t_0, \tag{58}$$

$$v(0,t) = v(1,t) = 0 \tag{59}$$

$$v(x,t_0) = v_{\text{in}}(x) \qquad \forall x \in \Omega, \tag{60}$$

where $v_{\text{in}}(x) = \frac{\partial u}{\partial x}(x,t_0)$.

Proposition 7. *The solution $v(x,t)$ of system (58)–(60) converges to a nonpositive function as $t \to \infty$.*

Proof. Let us denote by $v^+(x,t) = \max\{0, v(x,t)\}$. Since for each $t \geq t_0 > 0$, $v \in H^1(\Omega)$, we have $v^+ \in H^1(\Omega)$ (cf. [1], [7]). By taking the L^2 inner product of equation (58) with v^+, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (v^+)^2 dx + \int_0^1 ((v^+)_x)^2 dx + \int_0^1 F''(u)(v^+)^3 dx + \int_0^1 F'(u) v_x v^+ dx = 0.$$

Observe that

$$v_x v^+ = \frac{1}{2}((v^+)^2)_x.$$

Therefore,

$$\int_0^1 F'(u) v^+ v_x dx = -\frac{1}{2}\int_0^1 F''(u)(v^+)^3 dx.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (v^+)^2 dx + \int_0^1 ((v^+)_x)^2 dx + \frac{1}{2}\int_0^1 F''(u)(v^+)^3 dx = 0.$$

Recall that $F''(u) \geq 0$. Also notice that $v^+(0,t) = v^+(1,t) = 0$, then by applying the Poincaré inequality (see, e.g., [1], [7]) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (v^+)^2 dx + \pi^2 \int_0^1 (v^+)^2 dx \leq 0.$$

Thanks to Gronwall's inequality,

$$\|v^+(t)\|^2_{L^2(\Omega)} \leq \|v^+(t_0)\|^2_{L^2(\Omega)}\, e^{-2\pi^2(t-t_0)}. \tag{61}$$

Then, v^+ converges to zero as $t \to \infty$. Therefore, $v(x,t)$ converges to a nonpositive function as $t \to \infty$. □

Let $u(x,t)$ be the solution of the initial boundary value problem (1)–(3). We denote by

$$M(t) = \max_{0 \leq x \leq 1} u(x,t) \qquad \text{and} \qquad m(t) = \min_{0 \leq x \leq 1} u(x,t).$$

Notice that $u(x,t)$ satisfies *the maximum principle* (Proposition 2). Then, $M(t)$ is a nonincreasing function and $m(t)$ is a nondecreasing function. We denote by

$$\widetilde{M} = \lim_{t \to \infty} M(t) \qquad \text{and} \qquad \widetilde{m} = \lim_{t \to \infty} m(t). \tag{62}$$

Proposition 8. *Let $u(x,t)$ be the solution of the initial boundary value problem (1)–(3). Then*

$$\lim_{t \to \infty} (\widetilde{M} - u(0,t)) = \lim_{t \to \infty} (u(1,t) - \widetilde{m}) = 0. \tag{63}$$

Proof. Recall that $v(x,t) = u_x(x,t)$. Then

$$u(x,t) - u(0,t) = \int_0^x v(y,t)\,dy \le \int_0^1 v^+(y,t)\,dy.$$

Therefore,

$$M(t) - u(0,t) = \max_{0\le x\le 1} [u(x,t) - u(0,t)] \le$$

$$\le \int_0^1 v^+(y,t)dy \le \left(\int_0^1 (v^+(y,t))^2 dy\right)^{\frac{1}{2}}.$$

Since the right hand side goes to zero as $t \to \infty$, we conclude that

$$\lim_{t\to\infty} (\widetilde{M} - u(0,t)) = 0.$$

Similarly, we can show that

$$\lim_{t\to\infty} (u(1,t) - \widetilde{m}) = 0.$$

□

Theorem 9. *For any initial data $u_{\text{in}}(x) \in C^0(\overline{\Omega})$, the solution $u(x,t)$ of* (1)–(3) *converges to a steady state as $t \to \infty$. In other words, $u(x,t)$ converges to a constant function asymptotically in time.*

Proof. Let $u(x,t)$ be a solution of (1)–(3). It is obvious that the theorem is true when $\widetilde{M} = \widetilde{m}$, where $\widetilde{M}$ and $\widetilde{m}$ are as in (62). Therefore, we only show the theorem when $\widetilde{M} > \widetilde{m}$. Again we denote by $v = u_x$ and by

$$G(\varphi) = F(\varphi) - A_1\varphi - A_2, \tag{64}$$
$$w(x,t) = v(x,t) - G(u(x,t)), \tag{65}$$

where

$$A_1 = \frac{F(\widetilde{M}) - F(\widetilde{m})}{\widetilde{M} - \widetilde{m}} \tag{66}$$

$$A_2 = \frac{\widetilde{m}F(\widetilde{M}) - \widetilde{M}F(\widetilde{m})}{\widetilde{m} - \widetilde{M}} \tag{67}$$

where, again, $\widetilde{M}$ and $\widetilde{m}$ are as in (62). It is worth to mention that

$$G(\widetilde{M}) = G(\widetilde{m}) = 0. \tag{68}$$

By direct calculations, we get

$$\frac{\partial w}{\partial t} - \frac{\partial^2 w}{\partial x^2} + F'(u)\frac{\partial w}{\partial x} = 0. \tag{69}$$

By taking the L^2 inner product of equation (58) with v, and equation (1) with $G(u)G'(u)$, and by adding the resulting equations, we get

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [v^2 + G^2(u)]dx - \int_0^1 v_{xx}v dx + \int_0^1 F''(u)v^3 dx + \int_0^1 F'(u)v_x v dx -$$
$$- \int_0^1 u_{xx}G(u)G'(u)dx + \int_0^1 F'(u)u_x G(u)G'(u)dx = 0. \tag{70}$$

By integrating by parts, we reach

(1) $$-\int_0^1 v_{xx} v dx = \int_0^1 v_x^2\, dx,$$

(2) $$\int_0^1 F''(u) v^3\, dx = \int_0^1 G''(u) v\, v^2\, dx = -2 \int_0^1 G'(u) v v_x\, dx,$$

(3) $$\int_0^1 F'(u) v_x v dx = \frac{1}{2} \int_0^1 F'(u)(v^2)_x\, dx = -\frac{1}{2} \int_0^1 F''(u) v^3 dx,$$

(4) $$-\int_0^1 u_{xx} G(u) G'(u) dx = \int_0^1 u_x^2 (G'(u))^2\, dx + \int_0^1 u_x^2 G(u) G''(u)\, dx,$$
$$= \int_0^1 u_x^2 (G'(u))^2\, dx + \int_0^1 v^2 G(u) F''(u)\, dx,$$

(5) $$\int_0^1 F'(u) u_x G(u) G'(u) dx = \frac{1}{2} \int_0^1 \left[\left(F'(u)(G(u))^2 \right)_x - F''(u)(G(u))^2 v \right]\, dx =$$
$$= \frac{F'(u(1,t))[G(u(1,t))]^2 - F'(u(0,t))[G(u(0,t))]^2}{2} - \frac{1}{2} \int_0^1 F''(u)(G(u))^2 v\, dx$$

Therefore, equation (70) can be rewritten as

$$\frac{1}{2} \frac{d}{dt} \int_0^1 [v^2 + G^2(u)] dx + \int_0^1 w_x^2 dx - \frac{1}{2} \int_0^1 F''(u) v w^2 dx = \tag{71}$$
$$= \frac{F'(u(0,t))[G(u(0,t))]^2 - F'(u(1,t))[G(u(1,t))]^2}{2}.$$

Denote by

$$\alpha = \inf_{t_0 \le t < \infty} \int_0^1 w_x^2 dx.$$

We will show that $\alpha = 0$. Suppose that $\alpha > 0$. First, let us consider the following two terms:

$$F'(u(0,t))[G'(u(0,t))]^2 - F'(u(1,t))[G'(u(1,t))]^2$$

and

$$\frac{1}{2} \int_0^1 F''(u) v^+ w^2 dx.$$

By applying the Proposition 8, we obtain

$$\lim_{t \to \infty} \left(F'(u(0,t))[G'(u(0,t))]^2 - F'(u(1,t))[G'(u(1,t))]^2 \right)$$
$$= F'(\widetilde{M})[G'(\widetilde{M})]^2 - F'(\widetilde{m})[G'(\widetilde{m})]^2.$$

Thanks to equality (68)

$$\lim_{t \to \infty} \left(F'(u(0,t))[G'(u(0,t))]^2 - F'(u(1,t))[G'(u(1,t))]^2 \right) = 0. \tag{72}$$

Notice that

$$|F''(u)| \le \widetilde{F}(M_{\text{in}}).$$

Since $w(x,t)$ satisfies *the maximum principle* (see equation (69) and Proposition 2) and the regularity results of the previous section, then

$$\max_{0 \le x \le 1} |w(x,t)|^2 \le \max_{0 \le x \le 1} |w(x,t_0)|^2 \le (\|v(x,t_0)\|_{L^\infty} + \|G(u(x,t_0))\|_{L^\infty})^2 \le$$
$$\le \Big(K_3(t_0, M_{\text{in}}, K_1(t_0, M_{\text{in}})) + \widetilde{F}(M_{\text{in}}) + |A_1| M_{\text{in}} + |A_2| \Big)^2$$

where $\widetilde{F}, M_{\text{in}}, K_3, K_1$ are as in (4), (35), (30), (26), respectively. By applying the Proposition 7, we get

$$\lim_{t\to\infty}\frac{1}{2}\int_0^1 F''(u)v^+w^2dx = 0. \tag{73}$$

As a result of (72) and (73), there is a T, large enough, such that for any $t \geq T$

$$F'(u(0,t))[G'(u(0,t))]^2 - F'(u(1,t))[G'(u(1,t))]^2 \leq \alpha/4$$

and

$$\frac{1}{2}\int_0^1 F''(u)v^+w^2dx \leq \alpha/4.$$

Thus, from the above and equation (71) it follows that when $t \geq T$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [v^2 + G^2(u)]dx \leq -\alpha/2. \tag{74}$$

Since $\int_0^1 [v^2 + G^2(u)]dx \geq 0$, it would be impossible for inequality (74) to hold. Therefore, $\alpha = 0$. As a result, there is a sequence $\{t_j\}_{j=1}^{\infty}$, $t_j \to \infty$, such that

$$\lim_{j\to\infty}\int_0^1 w_x^2(x,t_j)dx = 0.$$

Applying the following version of Poincaré inequality

$$\left\|\phi(\cdot) - \int_0^1 \phi(x)dx\right\|_{L^\infty} \leq \left(\int_0^1 \phi_x^2(x)dx\right)^{1/2} \qquad \forall\phi \in H^1(\Omega),$$

we obtain

$$\max_{0\leq x\leq 1} w(x,t_j) - \min_{0\leq x\leq 1} w(x,t_j) \leq 2\left(\int_0^1 w_x^2(x,t_j)\,dx\right)^{1/2}.$$

Thus,

$$\lim_{t_j\to\infty}\left(\max_{0\leq x\leq 1} w(x,t_j) - \min_{0\leq x\leq 1} w(x,t_j)\right) = 0. \tag{75}$$

Since $w(x,t)$ satisfies *the maximum principle* (see equation (69) and Proposition 2), then

$$\left(\max_{0\leq x\leq 1} w(x,t) - \min_{0\leq x\leq 1} w(x,t)\right)$$

is a nonincreasing function of t. As a result of the above, We have

$$\lim_{t\to\infty}\left(\max_{0\leq x\leq 1} w(x,t) - \min_{0\leq x\leq 1} w(x,t)\right) = 0.$$

This shows that $w(x,t)$ converges to a constant function uniformly in x as $t \to \infty$. On the other hand, since

$$\lim_{t\to\infty} w(0,t) = -\lim_{t\to\infty} G(u(0,t)) = G(\widetilde{M}) = 0,$$

then,

$$\lim_{t\to\infty} w(x,t) = 0 \qquad \text{uniformly in } x. \tag{76}$$

Since $F \in C^2(\overline{\Omega})$ and

$$G(\widetilde{M}) = F(\widetilde{M}) - A_1\widetilde{M} - A_2 = 0,$$

there is a well–defined continuous function H such that

$$F(u) - A_1 u - A_2 = H(u)(u - \widetilde{M}).$$

Denote by $\widetilde{u} = u - \widetilde{M}$. Then, by the definition of w, we have

$$u_x - F(u) + A_1 u + A_2 = w,$$

and $\widetilde{u}$ satisfies

$$\widetilde{u}_x - H(u)\widetilde{u} = w.$$

Solving this linear equation with $\widetilde{u}$ is unknown gives

$$\widetilde{u} = u - \widetilde{M} = \left(u(0,t) - \widetilde{M} + \int_0^x w(x,t) e^{-\int_0^y H(u(z,t))dz} dy \right) e^{\int_0^x H(u(z,t))dz}.$$

Observe that by the mean value theorem

$$H(u(x,t)) = F'(\overline{u}(x,t)) + A_1, \tag{77}$$

where $\overline{u}(x,t)$ is between $u(x,t)$ and $\widetilde{M}$. From the maximum principle (Proposition 2, see also the statement of Theorem 3) we have

$$\|u(\cdot,t)\|_{L^\infty} \le M_{\text{in}} \qquad \text{and} \qquad |\widetilde{M}| \le M_{\text{in}},$$

and in particular we conclude that

$$\|u(\cdot,t)\|_{L^\infty} \le M_{\text{in}}.$$

Thus, from the above, (4) and (77) we conclude that

$$|H(u(x,t))| \le \widetilde{F}(M_{\text{in}}) + |A_1|.$$

Therefore,

$$e^{-\int_0^y H(u(z,t))dz} \le e^{\widetilde{F}(M_{\text{in}}) + |A_1|} \qquad \text{and} \qquad e^{\int_0^x H(u(z,t))dz} \le e^{\widetilde{F}(M_{\text{in}}) + |A_1|}.$$

Applying (63) and (76), we conclude that the right hand side of the above equation goes to zero as $t \to \infty$. Thus,

$$\lim_{t\to\infty} (u - \widetilde{M}) = 0.$$

Therefore, u converges to a constant as $t \to \infty$. □

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References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] A. Balogh and M. Krstic, *Global boundary stabilization and regularization of Burgers' equation*, http://www-mae.ucsd.edu/research/krstic/workshop.html, accessed September 17, 1999.

[3] J.M. Burgers, *Nonlinear Diffusion Equation,* Dordrecht, Netherlands: Reidel, 1974

[4] J.A. Burns and S. Kang, *A control problem for Burgers' equation with bounded input/output,* Nonlinear Dynamics, **2** (1991), 235–262.

[5] C.I. Byrnes and D.S. Gilliam, *Boundary feedback stabilization of a viscous Burgers' equation,* Computation and Control III: Proc. of the Third Bozeman Conference, Bozeman, Montana, August 5–11, 1992, K.I. Bowers, J. Lund (eds.), Birkhauser, Boston, 1993.

[6] H. Choi, R. Temam, P. Moin and J. Kim, *Feedback control for unsteady flow and its application to the stochastic Burgers equation,* J. Fluid Mech. **253** (1993), 509–543.

[7] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential equations of Second Order,* Springer–Verlag, 1983.

[8] A.T. Hill and E. Süli, *Dynamics of a nonlinear convection–diffusion equation in multidimensional bounded domains,* Proc. Roy. Soc. Edinburgh, **125A** (1995), 439–448.

[9] E. Hopf, *The partial differential equation* $u_t + uu_x = u_{xx}$, Comm. Pure Appl. Math. **3** (1950), 201–230.

[10] H.R. Jauslin, H.O. Kreiss and J. Moser, *On the forced Burgers equation with periodic boundary conditions,* Differential equations: La Pietra 1996 (Florence), 133–153, Proc. Sympos. Pure Math., **65**, Amer. Math. Soc., Providence, RI, 1999.

[11] F. John, *Partial Differential Equations,* Springer-Verlag, New York, 1982.

[12] R.H. Kraichnan, *Note on forced Burgers turbulence,* Physics of Fluids, **11** (1999), 3738–3742.

[13] G. Kreiss and H. Kreiss, *Convergence to steady state of solutions of Burgers' equation,* Appl. Numer. Math., **2** (1986), 161–179.

[14] M. Krstic, *On global stabilization of Burgers' equation by boundary control*, Systems & Control Letters, **37** (1999), 123–141.

[15] M. Krstic, T. Bewley and B. Bamieh, organizers: National Science Foundation Workshop on Control of Flows, http://www-mae.ucsd.edu/research/krstic/workshop.html, accessed September 17, 1999.

[16] H.V. Ly, K.D. Mease and E.S. Titi, *Distributed and boundary control of the viscous Burgers' equation,* Numer. Funct. and Optimiz., **18** (1997), 143–188.

[17] F.T.M. Nieuwstadt and J.A. Steketee, *Selected Papers of J.M. Burgers,* Kluwer, 1994

[18] M.H. Protter and H.F. Weinberger, *Maximum Principles in Differential Equations,* New York, Springer-Verlag, 1984.

[19] Z. She, E. Aurell and U. Frisch, *The inviscid Burgers equation with initial data of Brownian type,* Commun. Math. Phys., **148** (1992), 623–641.

[20] Ya.G. Sinai, *Statistics of shocks in solutions of inviscid Burgers equation,* Commun. Math. Phys., **148** (1992), 601–621.

[21] M.E. Taylor, *Differential Equations : Basic Theory,* New York, Springer, 1996.

[22] E. Weinan and E. Vanden Eijnden, *Another note on forced burgers turbulence,* Physics of Fluids, **12** (2000), 149–154.

[23] E. Weinan, K. Khanin, A. Mazel and Y. Sinai, *Probability distribution functions for the random forced burgers equation,* Phys. Rev. Lett., **78** (1997), 1904-1907.

[24] T.J. Zelenyak, *Stabilization of solutions of boundary value problems for a second order parabolic equation with one space variable,* Diff. Equat. **4** (1968), 17–22.

[25] T.J. Zelenyak, M.M. Lavrentiev Jr. and M.P. Vishnevskii, *Qualitative Theory of Parabolic Equations, Part 1*, VSP BV, 1997.

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