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On the Clark–α model of turbulence: global regularity and long-time dynamics

C. CAO*†, D. D. HOLM‡ and E. S. TITI§

†Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0323, USA; and Department of Mathematics, Florida International University, Miami, FL 33199, USA
‡Los Alamos National Laboratory, Los Alamos, NM 87545, USA; and Department of Mathematics, Imperial College London, SW7 2AZ, UK
§Department of Mathematics, and Department of Mechanical and Aerospace Engineering, University of California, Irvine, CA 92697-3875, USA; and Department of Computer Science and Applied Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel

In this paper we study a well-known three-dimensional turbulence model, the filtered Clark model, or Clark–α model. This is a large eddy simulation (LES) tensor-diffusivity model of turbulent flows with an explicit spatial filter of width α. We show the global well-posedness of this model with constant Navier–Stokes viscosity. Moreover, we establish the existence of a finite dimensional global attractor for this dissipative evolution system, and we provide an analytical estimate for its fractal and Hausdorff dimensions. Our estimate is proportional to $(L/l_d)^3$, where L is the integral spatial scale and l_d is the viscous dissipation length scale. This explicit bound is consistent with the physical estimate for the number of degrees of freedom based on heuristic arguments. Using semi-rigorous physical arguments we show that the inertial range of the energy spectrum for the Clark–α model has the usual $k^{-5/3}$ Kolmogorov power law for wave numbers $k\alpha \ll 1$ and k^{-3} decay power law for $k\alpha \gg 1$. This is an evidence that the Clark–α model parameterizes efficiently the large wave numbers within the inertial range, $k\alpha \gg 1$, so that they contain much less translational kinetic energy than their counterparts in the Navier–Stokes equations.

1. Introduction

This paper is devoted to the mathematical analysis of a large eddy simulation (LES) model of turbulent flows with explicit filtering. We consider the 'filtered Clark model,' also known as the 'Clark–α model' [1]. This is a non-linear LES model of turbulence consisting of the tensor-diffusivity model of Leonard [2], filtered by inversion of the Helmholtz operator with width α. As pointed out by Winckelmans *et al.* [3], this model is generic: for all regular symmetric filters that have a non-zero second moment, this form is found as the first term of the reconstruction series for the filtered-scale stress. Thus, the Clark–α model we consider is generic and corresponds to significant, yet perhaps incomplete, reconstruction of the filtered-scale stress. Such an explicitly filtered tensor-diffusivity model was used by Vreman *et al.* [4, 5] with good success, in combination with a dynamic Smagorinsky term as a 'non-linear mixed model.' In their work, the Smagorinsky term was posed to model the truncation effects of subgrid-scale stress and incomplete reconstruction of the filtered-scale stress. Vreman *et al.* [4, 5] referred to their model as 'a dynamic version of the mixed Clark model', while

*Corresponding author. E-mail: ccao@math.unl.edu

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Winckelmans *et al.* [3] wrote that they preferred the term ‘tensor-diffusivity model’ for such models. The results given in [3] supported the view that the explicitly filtered tensor-diffusivity (or Clark) model may well suffice for practical reconstruction of the filtered-scale stress, *without* introducing a Smagorinsky term.

Section 2 is devoted to the precise definition of the Clark–α model and to discussions of the properties of its non-linear terms. Section 3 establishes the main results of global (in time) regularity for the Clark–α model, that is, global existence, uniqueness and continuous dependence of the initial data of the solutions to this three-dimensional (3-D) model of turbulence. Section 4 shows that the ‘Clark–α turbulence’ model possesses a global attractor and gives an upper bound on its Hausdorff and fractal dimensions. This upper bound is discovered to scale as $(L/l_d)^3$ – the cube of the integral scale L divided by the dissipation length $l_d = (\nu^3/\epsilon)^{1/4}$. Finally, the translational kinetic energy spectrum $E(k)$ for the Helmholtz-filtered Clark–α model is also shown in section 4 to pass from $k^{-5/3}$ for smaller wave numbers ($k\alpha \ll 1$), to k^{-3} for larger wave numbers ($k\alpha \gg 1$). It is worth mentioning that the results presented here are similar to those already established for the Navier–Stokes–α (NS–α) model [6, 7] (also known as the 3-D viscous Camassa–Holm equations, or the Lagrangian averaged NS–α model). Furthermore, when viewed as a closure model of turbulence for the Reynolds averaged equations in turbulent channels and pipes the Clark–α model gives exactly the same reduced equations in channels and pipes as the NS–α model. Comparing the solutions of those reduced equations in pipes and channels to the empirical data has already given excellent agreement (see [8–11]). Therefore, in this regard, one can assert that the Clark–α model is as successful as the NS–α sub-grid model of turbulence.

2. Preliminaries and notations

Let $\Omega = [0, 2\pi L]^3$. The so-called Clark–α turbulence model (cf. [1]) of viscous incompressible flows in a domain Ω subject to the periodic boundary condition reads:

$$\partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p - \nabla\cdot(\mathcal{H}^{-1}(\alpha^2\nabla u\cdot\nabla u^T)) = f, \tag{1}$$

$$\nabla\cdot u = 0, \tag{2}$$

$$u(0) = u_0, \tag{3}$$

where u represents the unknown ‘averaged/filtered’ fluid velocity vector, and p is the unknown ‘averaged/filtered’ pressure scalar; $\nu > 0$ is the constant kinematic viscosity, α is a length scale parameter which represents the width of the filter. The body forcing term, f, which is time independent, and the initial velocity, u_0, are given. The operator $\mathcal{H}$ is the Helmholtz operator, defined as

$$\mathcal{H}u = u - \alpha^2\Delta u,$$

subject to periodic boundary condition, and the tensor $(\nabla u\cdot\nabla u^T)$ is given by

$$(\nabla u\cdot\nabla u^T)_{ij} = \nabla u_i\cdot\nabla u_j = \sum_{k=1}^{3}\left(\partial_{x_k}u_i\right)\left(\partial_{x_k}u_j\right).$$

From now on, the Einstein’s summation convention will be used, for example,

$$\left(\partial_{x_k}u_i\right)\left(\partial_{x_k}u_j\right) = \sum_{k=1}^{3}\left(\partial_{x_k}u_i\right)\left(\partial_{x_k}u_j\right).$$

Let $v = \mathcal{H}u = u - \alpha^2 \Delta u$. Then, the above system is equivalent to

$$\partial_t v - \nu \Delta v + (u \cdot \nabla) v + (v \cdot \nabla) u - (u \cdot \nabla) u - \alpha^2 \nabla \cdot (\nabla u \cdot \nabla u^T) + \nabla q = g, \quad (4)$$

$$\nabla \cdot u = \nabla \cdot v = 0, \quad (5)$$

$$v(0) = v_0 = \mathcal{H} u_0, \quad (6)$$

where $q = \mathcal{H} p$ and $g = \mathcal{H} f$.

2.1 *Notations*

Let $L^p(\Omega)$ and $H^m(\Omega)$ denote the usual L^p Lebesgue spaces and Sobolev spaces, respectively (cf. [12]). We will denote by $|\cdot|$ the L^2- norm, and $(\cdot,\cdot)$ the L^2- inner product. Let $\mathcal{F}$ be the function space which consists of all the vector trigonometric polynomials. We set

$$\mathcal{V} = \left\{ \phi \in \mathcal{F} : \nabla \cdot \phi = 0 \qquad \text{and} \qquad \int \phi = 0 \right\}.$$

The spaces H, V_1 and V_2 will denote the closures of $\mathcal{V}$ in L^2, H^1, and H^2, respectively. Let $P_\sigma \colon L^2 \to H$ be the orthogonal projection, and let $A = -P_\sigma \Delta$ be the Stokes operator subject to the periodic boundary conditions. It is well known that A^{-1} is a self-adjoint positive compact operator in H and $\mathcal{D}(A) = V_2$ (cf. [13–16]). Let $0 < \lambda_1 = L^{-2} \le \lambda_2 \le \cdots$ be the eigenvalues of A, repeated according to their multiplicities. It is well known that a constant $C_1 > 0$ exists such that

$$\frac{j^{2/3}}{C_1} \le \frac{\lambda_j}{\lambda_1} \le C_1 j^{2/3}, \quad (7)$$

for $j = 1, 2, \ldots$ (see, for example, reference [13]).

For $w_1, w_2 \in \mathcal{V}$, we define

$$B(w_1, w_2) = P_\sigma \left((w_1 \cdot \nabla) w_2 \right).$$

The bilinear form B has the following properties (cf. [13, 16]).

PROPOSITION 1 *The bilinear form B can be extended as a continuous map $B\colon V_1 \times V_1 \to V_1'$, where V_1' is the dual space of V_1. Furthermore,*

$$\langle B(w_1, w_2), w_3 \rangle_{V_1'} = -\langle B(w_1, w_3), w_2 \rangle_{V_1'}$$

for every $w_1, w_2, w_3 \in V_1$.

PROPOSITION 2 *Let $u \in H^3 \cap V_1$, $v = \mathcal{H}u = u - \alpha^2 \Delta u$. Then, the bilinear operators $B(u, v)$, $B(v, u)$ and $\sum_{j=1}^{3} B(\partial_{x_j} u, \partial_{x_j} u)$ are well defined with values in V_1'.*

The following properties can be established by integration by parts.

PROPOSITION 3 *Let $u \in H^3 \cap V_1$, $v = \mathcal{H}u = u - \alpha^2 \Delta u$. Then,*

(i) $\langle B(u, u), u \rangle_{V_1'} = 0$,
(ii) $\langle B(v, u), u \rangle_{V_1'} = 0$,
(iii) $\langle B(u, v), u \rangle_{V_1'} - \alpha^2 \sum_{j=1}^{3} \langle B(\partial_{x_j} u, \partial_{x_j} u), u \rangle_{V_1'} = 0$.

As for the case of the Navier–Stokes equations (cf. [13, 16]) by applying the operator P_σ, we get the following system which is equivalent to the system (1)–(3)

$$\frac{du}{dt} + \nu Au + B(u,u) + \alpha^2 \mathcal{H}^{-1} B\big(\partial_{x_j} u, \partial_{x_j} u\big) = f, \tag{8}$$

$$u(0) = u_0, \tag{9}$$

and the following system which is equivalent to the system (4)–(6)

$$\frac{dv}{dt} + \nu Av + B(u,v) + B(v,u) - B(u,u) - \alpha^2 B\big(\partial_{x_j} u, \partial_{x_j} u\big) = g, \tag{10}$$

$$v(0) = v_0. \tag{11}$$

Definition 4 Let T be any fixed positive number and $f \in V_1$. We call u a regular solution of (8)–(9) as an equation in V_1 on $[0, T]$ if u satisfies the system (8)–(9) and

$$u \in C_w([0,T], V_2) \cap L^2([0,T], H^3(\Omega)), \qquad \text{and} \qquad \frac{du}{dt} \in L^2([0,T], V_1).$$

Here $C_w([0,T], V_2)$ is the functional space of all weakly continuous functions from $[0, T]$ to V_2.

For convenience, we recall the following Sobolev and Ladyzhenskaya's inequalities in $\mathbb{R}^3$

$$\|u\|_{L^3} \le C_0 \|u\|_{L^2}^{1/2} \|u\|_{H^1}^{1/2}, \tag{12}$$

$$\|u\|_{L^4} \le C_0 \|u\|_{L^2}^{1/4} \|u\|_{H^1}^{3/4}, \tag{13}$$

$$\|u\|_{L^6} \le C_0 \|u\|_{H^1}, \tag{14}$$

for every $u \in V_1$, and the Agmon's inequality in $\mathbb{R}^3$

$$\|u\|_{L^\infty} \le C_0 \|u\|_{H^1}^{1/2} \|u\|_{H^2}^{1/2}, \tag{15}$$

for every $u \in V_2$. Here C_0 is a universal constant. Also, notice that

$$|\mathcal{H}u|^2 \le \big(\lambda_1^{-1} + \alpha^2\big)(|\nabla u|^2 + \alpha^2 |\Delta u|^2). \tag{16}$$

For this model the quantity $(u, v) = |u|^2 + \alpha^2 |\nabla u|^2$ will be called the energy of the system, while the quantity $(u, u) = |u|^2$ will be called the translational energy of the system.

3. Existence and uniqueness

3.1 H^1 estimate

The estimates presented here are formal. They can be made rigorous by establishing them first for a Galerkin approximating scheme and then passing to the limit by using an Aubin compactness theorem (see, for example, [13, 16–17]). By taking the V_1' action of equation (8) with $v = \mathcal{H}u$ and using Proposition 3 and Lemma 1.2 in Chapter III of [16], one finds the energy balance satisfies:

$$\frac{1}{2}\frac{d(|u|^2+\alpha^2|\nabla u|^2)}{dt}+\nu(|\nabla u|^2+\alpha^2|\Delta u|^2)+\langle B(u,u),\mathcal{H}u\rangle_{V_1'}-\alpha^2\left\langle B\big(\partial_{x_j}u,\partial_{x_j}u\big),u\right\rangle_{V_1'}$$
$$=\langle f,\mathcal{H}u\rangle_{V_1'}. \tag{17}$$

By using part (iii) of Proposition 3 we obtain

$$\frac{1}{2}\frac{d(|u|^2+\alpha^2|\nabla u|^2)}{dt}+\nu(|\nabla u|^2+\alpha^2|\Delta u|^2)=(f,\mathcal{H}u) \tag{18}$$
$$\le |f||u|+\alpha^2|\nabla f||\nabla u| \tag{19}$$
$$\le \frac{1}{2\nu\lambda_1}(|f|^2+\alpha^2|\nabla f|^2)+\frac{\nu\lambda_1}{2}(|u|^2+\alpha^2|\nabla u|^2) \tag{20}$$
$$\le \frac{1}{2\nu\lambda_1}(|f|^2+\alpha^2|\nabla f|^2)+\frac{\nu}{2}(|\nabla u|^2+\alpha^2|\Delta u|^2). \tag{21}$$

As a result, we reach

$$\frac{d(|u|^2+\alpha^2|\nabla u|^2)}{dt}+\nu(|\nabla u|^2+\alpha^2|\Delta u|^2)\le\frac{|f|^2+\alpha^2|\nabla f|^2}{\nu\lambda_1}. \tag{22}$$

Thanks to the Gronwall inequality, we get

$$|u|^2+\alpha^2|\nabla u|^2\le K_1(\alpha,\nu,t), \tag{23}$$

where

$$K_1(\alpha,\nu,t)=e^{-\nu\lambda_1 t}(|u_0|^2+\alpha^2|\nabla u_0|^2)+\frac{|f|^2+\alpha^2|\nabla f|^2}{\nu^2\lambda_1^2}. \tag{24}$$

Moreover, by equation (22), for each $r>0$, we have

$$\nu\int_t^{t+r}(|\nabla u|^2+\alpha^2|\Delta u|^2)\le\frac{(|f|^2+\alpha^2|\nabla f|^2)r}{\nu\lambda_1}+K_1(\alpha,\nu,t). \tag{25}$$

Consequently,

$$\int_0^t(|\nabla u|^2+\alpha^2|\Delta u|^2)\le\frac{(|f|^2+\alpha^2|\nabla f|^2)t}{\nu^2\lambda_1}+\frac{1}{\nu}(|u_0|^2+\alpha^2|\nabla u_0|^2). \tag{26}$$

3.2 H^2 *estimate*

Again, the estimates presented here are formal. As above, they can be made rigorous by establishing them first for a Galerkin approximating scheme and then passing to the limit. By taking the V_1' action of equation (10) with $v=\mathcal{H}u$, one finds

$$\frac{1}{2}\frac{d|v|^2}{dt}+\nu|\nabla v|^2+\left\langle B(u,v)+B(v,u)-B(u,u)-\alpha^2B\big(\partial_{x_j}u,\partial_{x_j}u\big),v\right\rangle_{V_1'}=\langle g,v\rangle_{V_1'}. \tag{27}$$

Notice that

$$\left|\left\langle B(u,v)+B(v,u)-B(u,u)-\alpha^2B\big(\partial_{x_j}u,\partial_{x_j}u\big),v\right\rangle_{V_1'}\right|$$
$$=\left|\left\langle B(v,u)-B(u,u)-\alpha^2B\big(\partial_{x_j}u,\partial_{x_j}u\big),v\right\rangle_{V_1'}\right|$$
$$\le|\nabla u|\|v\|_{L^4}^2+|v|\|u\|_{L^3}\|u\|_{L^6}+\alpha^2|\nabla v|\|\nabla u\|_{L^4}^2$$
$$\le C_0^2|\nabla u||v|^{1/2}|\nabla v|^{3/2}+C_0^2|v||u|^{1/2}|\nabla u|^{3/2}+C_0^2\alpha^2|\nabla v||\nabla u|^{1/2}|\Delta u|^{3/2}.$$

Also,

$$\big|\langle g, v\rangle_{V_1'} = \langle \mathcal{H} f, v\rangle_{V_1'}\big| = \big|\langle f, v\rangle_{V_1'} - \alpha^2 \langle \Delta f, v\rangle_{V_1'}\big| \\ \leq |f||v| + \alpha^2|\nabla f||\nabla v|.$$

By Cauchy–Schwarz and Young's inequalities we reach

$$\begin{aligned} &\frac{d|v|^2}{dt} + \frac{\nu}{4}|\nabla v|^2 \\ &\leq |f|^2 + |v|^2 + \alpha^4|\nabla f|^2 + \frac{2C_0^8|\nabla u|^4|v|^2}{\nu^3} + 8C_0^2|v||u|^{1/2}|\nabla u|^{3/2} + \frac{8C_0^4\alpha^4|\nabla u||\Delta u|^3}{\nu} \\ &\leq |f|^2 + \alpha^4|\nabla f|^2 + \left[1 + \frac{2C_0^8|\nabla u|^4}{\nu^3} + \frac{8C_0^2\lambda_1^{1/2}|u|^{1/2}|\nabla u|^{1/2}}{\alpha^2}\frac{8C_0^4|\nabla u||\Delta u|}{\nu}\right]|v|^2. \end{aligned}$$

By (23), (26), and Gronwall inequality we get

$$|v(t)|^2 \leq K_2(\alpha, \nu, t)(|v|^2(s) + t(|f|^2 + \alpha^4|\nabla f|^2)), \tag{28}$$

where

$$\begin{aligned} K_2(\alpha, \nu, t) = \exp\Bigg(&\left[1 + \frac{2C_0^8(K_1(\alpha, \nu, 0))^2}{\nu^3\alpha^4} + \frac{8C_0^2\lambda_1^{3/4}(K_1(\alpha, \nu, 0))^{1/2}}{\alpha^3} + \frac{8C_0^4|f|^2}{\nu^3\alpha^2\lambda_1^{1/2}}\right]t \\ &+ \frac{8C_0^4|u_0|^2 + \alpha^2|\nabla u_0|^2}{\nu^2}\Bigg). \end{aligned} \tag{29}$$

In particular,

$$|v(t)|^2 \leq K_2(\alpha, \nu, t)(|v_0|^2 + t(|f|^2 + \alpha^4|\nabla f|^2)). \tag{30}$$

Moreover,

$$\frac{\nu}{4}\int_0^t |\nabla v|^2 \leq (|f|^2 + \alpha^4|\nabla f|^2)t + K_2(\alpha, \nu, t)(|v_0|^2 + t(|f|^2 + \alpha^4|\nabla f|^2)). \tag{31}$$

These analytical estimates lead to the following.

THEOREM 5 *Let $u_0 \in V_1$ and $f \in V_1$. Then there is a unique regular solution to system* (8)–(9) *for all $t > 0$. Furthermore, this solution depends continuously on the initial data in the sense that it will be specified in the proof below.*

Proof One can establish the existence of a regular solution by applying the standard Galerkin approximation procedure together with the a priori estimates (23), (26), (30) and (31) (see, for example, [13–16]). Here, we shall only show the uniqueness of regular solution to the system (8)–(9) and the continuous dependence of solutions on the initial data. Suppose that u_1 and u_2 are two regular solutions to the system (8)–(9), and let $w = u_1 - u_2$. Then the difference w satisfies

$$\begin{aligned} &\frac{dw}{dt} + \nu Aw + B(w, u_1) + B(u_2, w) + \alpha^2\mathcal{H}^{-1}B\big(\partial_{x_j}w, \partial_{x_j}u_1\big) \\ &\quad + \alpha^2\mathcal{H}^{-1}B\big(\partial_{x_j}u_2, \partial_{x_j}w\big) = 0, \end{aligned} \tag{32}$$

$$w(0) = 0. \tag{33}$$

By taking the V_1' action of equation (32) with $\mathcal{H}w$ and applying Lemma 1.2 in Chapter III of [16], one finds

$$\begin{aligned}
&\frac{1}{2}\frac{d(|w|^2+\alpha^2|\nabla w|^2)}{dt}+\nu(|\nabla w|^2+\alpha^2|\Delta w|^2)\\
&\quad=-\left\langle B(w,u_1)+B(u_2,w)+\alpha^2\mathcal{H}^{-1}B\left(\partial_{x_j}w,\partial_{x_j}u_1\right)+\alpha^2\mathcal{H}^{-1}B\left(\partial_{x_j}u_2,\partial_{x_j}w\right),\mathcal{H}w\right\rangle_{V_1'}\\
&\quad\le|\mathcal{H}w|(\|w\|_{L^\infty}|\nabla u_1|+\|u_2\|_{L^\infty}|\nabla w|)+\alpha^2(|\nabla u_1|+|\nabla u_2|)\|\nabla w\|_{L^4}^2\\
&\quad\le C_0|\mathcal{H}w|(|\nabla w|^{1/2}|\Delta w|^{1/2}|\nabla u_1|+|\Delta u_2||\nabla w|)+\alpha^2C_0(|\nabla u_1|+|\nabla u_2|)|\nabla w|^{1/2}|\Delta w|^{3/2}.
\end{aligned}$$

By Cauchy–Schwarz inequality, we obtain

$$\frac{1}{2}\frac{d(|w|^2+\alpha^2|\nabla w|^2)}{dt}\le C(|\Delta u_2|^2+|\nabla u_1|^4+|\nabla u_2|^4)|\nabla w|^2,$$

where C is a constant. Thanks to the Gronwall inequality, we get

$$|w(t)|^2+\alpha^2|\nabla w(t)|^2\le\exp\left(\frac{C}{\alpha^2}\int_0^t(|\Delta u_2|^2+|\nabla u_1|^4+|\nabla u_2|^4)\right)(|w(0)|^2+\alpha^2|\nabla w(0)|^2).$$

Owing to (23), (26), (30), and (31), the above implies continuous dependence on initial data. In particular,

$$w(t)=0\qquad\text{when}\qquad w(0)=0.$$

Therefore, the regular solution is unique. ■

Remark 1. In order to be able to estimate the dimension of the global attractor it is required that the solution is differentiable, in the appropriate norms, with respect to the initial data. Following similar energy methods to the ones introduced above one can establish the required differentiability (see, for example, [13, 17, 18]).

Remark 2. Following the work of Foias and Temam [19] (see also [20]) one can show that the solution of (4)–(6) instantanously becomes analytic, provided that the forcing term belongs to the appropriate Gevrey class of regularity.

4. Global attractors and their dimensions

In this section we show the existence of the global attractor. Moreover, we provide an upper bound to its fractal and Hausdorff dimension. Also, we consider the energy spectrum for the Clark–α model.

4.1 *Global attractor*

Denote by $u(t)=S(t)u_0$ the solution of the system (8)–(9) with initial data u_0. As a result of Theorem 5, one can show that

$$u(t)=S(t)u_0\in V_1\qquad\text{for all}\qquad u_0\in V_1,\,t\ge0,$$

and

$$u(t)=S(t)u_0\in V_2\qquad\text{for all}\qquad u_0\in V_2,\,t\ge0.$$

Since, in this section, we only consider the long-time behaviour of solutions of the system (8)–(9), by (26), (28), and Theorem 5, we conclude that $u(t) \in L^{\infty}_{\text{loc}}((0, S], V_2)$ for every $u(0) \in V_1$ and any $S > 0$.

THEOREM 6 *Suppose that* $f \in V_1$. *Then, there is a compact global attractor* $\mathcal{A} \subset V_1$ *for the system* (8)–(9). *Moreover,* $\mathcal{A}$ *has finite Hausdorff and fractal dimensions.*

Proof First, let us show that there is an absorbing ball in V_1 and V_2. Let u be the solution of the system (8)–(9) with initial data $u_0 \in V_1$ and $|u_0|^2 + \alpha^2|\nabla u_0|^2 \le \rho$. By (22), we have

$$|u(t)|^2 + \alpha^2|\nabla u(t)|^2 \le e^{-\nu\lambda_1 t}(|u_0|^2 + \alpha^2|\nabla u_0|^2) + \frac{|f|^2 + \alpha^2|\nabla f|^2}{\nu^2\lambda_1^2}$$
$$\le e^{-\nu\lambda_1 t}\rho^2 + \frac{|f|^2 + \alpha^2|\nabla f|^2}{\nu^2\lambda_1^2}.$$

As a result of the above, when t is large enough such that

$$e^{-\nu\lambda_1 t}\rho^2 \le \frac{|f|^2 + \alpha^2|\nabla f|^2}{\nu^2\lambda_1^2},$$

we have

$$|u(t)|^2 + \alpha^2|\nabla u(t)|^2 \le R_a(\alpha, \nu, f), \tag{34}$$

where

$$R_a(\alpha, \nu, f) = 2\frac{|f|^2 + \alpha^2|\nabla f|^2}{\nu^2\lambda_1^2}. \tag{35}$$

In particular,

$$\limsup_{t\to\infty}(|u(t)|^2 + \alpha^2|\nabla u(t)|^2) \le 2\frac{|f|^2 + \alpha^2|\nabla f|^2}{\nu^2\lambda_1^2}.$$

Therefore, system (8)–(9) has an absorbing ball $\mathcal{B}$ in V_1 with radius $R_a(\alpha, \nu, f)$.

Next, we show that there is an absorbing ball in V_2. First, by (34) and (25) we have

$$\int_t^{t+r}(|\nabla u|^2 + \alpha^2|\Delta u|^2) \le \nu\int_t^{t+r}(|\nabla u|^2 + \alpha^2|\Delta u|^2) \le \frac{|f|^2 + \alpha^2|\nabla f|^2(2 + r\nu\lambda_1)}{\nu^3\lambda_1^2}. \tag{36}$$

By applying the uniform Gronwall inequality (cf., for example, [17], p. 89) and (36) and (26), we obtain, when t is large enough,

$$|\mathcal{H}u(t)|^2 = |v(t)|^2 \le R_v(r, \alpha, \nu, f), \tag{37}$$

where

$$R_v(r, \alpha, \nu, f) = \exp\left(\frac{C_0^{1/2}}{\alpha^2}(R_a(\alpha, \nu, f))^{1/2} + \frac{2r^{1/3}}{\alpha^{8/3}\nu^{4/3}}\left(\frac{|f|^2 + \alpha^2|\nabla f|^2(2 + r\nu\lambda_1)}{\nu^3\lambda_1^2}\right)^{2/3}\right)$$
$$\times\left(\frac{(|f|^2 + \alpha^2|\nabla f|^2)(2 + r\nu\lambda_1)}{\nu^3\lambda_1^2 r} + \frac{r}{\nu}|\nabla f|^2\right), \tag{38}$$

and $r > 0$ is fixed. Therefore, we conclude there is an absorbing ball $\mathcal{B}$ in V_2 with radius $R_v(r, \alpha, \nu, f)$. Thanks to Rellich Lemma (see, for example, [12]), the operator $S(t)$ is a compact operator from V_1 to itself. Following the standard procedure (cf., for example, [13, 17, 18, 21, 22] for details), one can prove that there is a global attractor

$$\mathcal{A} = \cap_{s>0}(\cup_{t>s} S(t)\mathcal{B}) \subset V_2.$$

Moreover, $\mathcal{A}$ is compact in V_1.

By using the estimates (34) and (36), applying Lemma 4 and Lemma 5 in [6], and following the proof of Theorem 6 in [6] line by line, we obtain the Hausdorff and fractal dimensions of the attractor $\mathcal{A}$

$$d_H(\mathcal{A}) \leq d_F(\mathcal{A}) \leq C \max\left\{ \frac{(|f|^2 + \alpha^2 |\nabla f|^2)^{2/3}}{\alpha^{4/3}\nu^{8/3}\lambda_1^{5/3}}, \frac{(|f|^2 + \alpha^2 |\nabla f|^2)^{3/4}}{\alpha^{3/4}\nu^3\lambda_1^{3/2}} \right\}.$$

Moreover, following [6], we define the mean rate of dissipation of energy,

$$\epsilon = \sup_{u_0\in\mathcal{A}} \limsup_{T\to\infty} \frac{\nu}{T}\int_0^T \left(\|\nabla u(t)\|_{L^2}^2 + \alpha^2 \|\Delta u(t)\|_{L^2}^2\right), \tag{39}$$

where $u(t)$ is the solution corresponding to the initial datum u_0. By analogy, we define the Kolmogorov dissipation length scale corresponding to the system (1)–(3),

$$l_d = \left(\frac{\nu^3}{\epsilon}\right)^{1/4}.$$

Hence, again, following the proof of Theorem 7 in reference [6], we have

$$d_H(\mathcal{A}) \leq d_F(\mathcal{A}) \leq C\left(\frac{L}{\alpha}\right)^{3/4}\left(\frac{L}{l_d}\right)^3.$$

We refer the reader to [6] for more details. ∎

5. Energy spectra

Following [7] (see also [23, 24]), we set

$$\hat{u}_k = \frac{1}{(2\pi L)^3}\int_\Omega u(x)e^{-ik\cdot x}dx,$$

$$\hat{v}_k = \frac{1}{(2\pi L)^3}\int_\Omega v(x)e^{-ik\cdot x}dx,$$

$$u_k = \sum_{k\leq|j|<2k} \hat{u}_j e^{ij\cdot x},$$

$$v_k = \sum_{k\leq|j|<2k} \hat{v}_j e^{ij\cdot x},$$

$$u_k^< = \sum_{j<k} u_j, \qquad v_k^< = \sum_{j<k} v_j$$

$$u_k^> = \sum_{2k\leq j} u_j, \qquad v_k^> = \sum_{2k\leq j} v_j.$$

The energy balance equation for the Clark–α model for an eddy of the size k^{-1} is

$$\frac{1}{2}\frac{d}{dt}(u_k, v_k) + \nu(-\Delta u_k, v_k) = T_k - T_{2k}, \tag{40}$$

where the right-hand side is the energy flux and

$$T_k = -(B(u_k^<, u_k^<), v_k) - \alpha^2(B(\nabla u_k^<, \nabla u_k^<), u_k) +$$
$$+ (B(u_k + u_k^>, u_k + u_k^>), v_k^<) + \alpha^2(B(\nabla u_k + \nabla u_k^>, \nabla u_k + \nabla u_k^>), u_k^<).$$

Taking an ensemble average (long time average) of (40) we get

$$\langle \nu(-\Delta u_k, v_k)\rangle = \langle T_k\rangle - \langle T_{2k}\rangle \,. \tag{41}$$

Let

$$E_\alpha(k) = (1 + \alpha^2 \mid k \mid^2) \sum_{|j|=k} |\hat{u}_j|^2.$$

Then (41) can be written as

$$\nu k^3 E_\alpha(k) \sim \int_k^{2k} k^2 E_\alpha(k) dk \sim \langle T_k\rangle - \langle T_{2k}\rangle \,.$$

For wave numbers k within the inertial range it is assumed that there is no energy disspiation, hence we have $\nu k^3 E_\alpha(k) \ll \langle T_k\rangle$, and we also have $\langle T_k\rangle \approx \langle T_{2k}\rangle$, because there is no leakage of energy. One can consider three possible scales for the average velocity on an eddy of length size k^{-1}. Namely,

$$U_k^0 = \left\langle \frac{1}{L^3}\int_\Omega |v_k \mid^2 dx\right\rangle^{1/2} \sim (k(1+\alpha^2k^2)E_\alpha(k))^{1/2},$$
$$U_k^1 = \left\langle \frac{1}{L^3}\int_\Omega u_k \cdot v_k dx\right\rangle^{1/2} \sim (kE_\alpha(k))^{1/2},$$
$$U_k^2 = \left\langle \frac{1}{L^3}\int_\Omega |u_k|^2 dx\right\rangle^{1/2} \sim \left(\frac{kE_\alpha(k)}{1+\alpha^2k^2}\right)^{1/2}.$$

The corresponding turnover time τ_k for an eddy of the size k^{-1} will be (cf. [25])

$$\tau_k^n \sim \frac{1}{kU_k^n} = \frac{(1+\alpha^2k^2)^{(n-1)/2}}{k^{3/2}(E_\alpha(k))^{1/2}}, \qquad n = 0, 1, 2.$$

The energy dissipation rate ϵ (39) is

$$\epsilon \sim \frac{1}{\tau_k^n}\int_k^{2k} E_\alpha(k)dk \sim \frac{k^{5/2}(E_\alpha(k))^{3/2}}{(1+\alpha^2k^2)^{(n-1)/2}}.$$

As a result, we have

$$E_\alpha(k) \sim \frac{\epsilon^{2/3}(1+\alpha^2k^2)^{(n-1)/3}}{k^{5/3}}.$$

Therefore, the translational kinetic energy spectrum $E(k) = \sum_{|j|=k} |\hat{u}_j|^2$ is given by

$$E(k) \equiv \frac{E_\alpha(k)}{1+\alpha^2k^2} \sim \begin{cases} \dfrac{\epsilon_\alpha^{2/3}}{k^{5/3}}, & \text{when } k\alpha \ll 1, \\[2ex] \dfrac{\epsilon_\alpha^{2/3}}{\alpha^{2(4-n)/3}k^{(13-2n)/3}}, & \text{when } k\alpha \gg 1. \end{cases}$$

We refer the reader to [7] and [26] for more details and discussions of the implication of the energy spectrum $k^{-(13-2n)/3}$ for the larger wavenumbers ($1 \ll k\alpha$).

Remark 3. It is worth stressing that it is not clear from the model which one of the U_k^n, for $n = 0, 1, 2$, is the right average velocity for an eddy of the size k^{-1}. Consequently, it is not evident what would be the correct energy spectra in the sub-range, $k\alpha \gg 1$, of the inertial range. This will be a subject of future research.

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