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SUFFICIENT CONDITIONS FOR THE REGULARITY TO THE 3*D* NAVIER–STOKES EQUATIONS

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Abstract. In this paper we consider the three–dimensional Navier–Stokes equations subject to periodic boundary conditions or in the whole space. We provide sufficient conditions, in terms of one direction derivative of the velocity field, namely, u_z, for the regularity of strong solutions to the three-dimensional Navier–Stokes equations.

1. **Introduction.** The three-dimensional Navier–Stokes equations (NSE) of viscous incompressible fluid read:

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u\cdot\nabla)u + \nabla p = 0, \tag{1}$$

$$\nabla \cdot u = 0, \tag{2}$$

$$u(x,y,z,0) = u_0(x,y,z), \tag{3}$$

where u, the velocity field, and p, the pressure, are the unknowns, and $\nu > 0$, the viscosity, is given. We equip the system (1)–(3) with periodic boundary conditions with period 1. Namely,

$$u(x+1,y,z,t) = u(x,y+1,z,t) = u(x,y,z+1,t) = u(x,y,z,t). \tag{4}$$

Because of the periodic boundary conditions we consider here the base domain $\Omega = (0,1)^3$. We emphasize, however, that one can apply our proof almost line by line to obtain same results for the three–dimensional Navier–Stokes equations in the whole space $\mathbb{R}^3$.

We denote by $L^q(\Omega)$ and $H^m(\Omega)$ the usual L^q-Lebesgue and Sobolev spaces, respectively (cf. [1]). We denote by

$$\|\phi\|_q = \left(\int_\Omega |\phi|^q \; dxdydz\right)^{\frac{1}{q}}, \qquad \text{for every } \phi \in L^q(\Omega). \tag{5}$$

We set

$$\mathcal{V} \;=\; \{\phi : \text{the } 3D \text{ vector valued trigonometric polynomials with } \nabla\cdot\phi = 0\},$$

which will form the space of test functions. Let H and V be the closure spaces of $\mathcal{V}$ in $L^2(\Omega)$ under L^2-topology, and in $H^1(\Omega)$ under H^1-topology, respectively.

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Let $u_0 \in H$, we say u is a Leray–Hopf weak solution to the system (1)–(3) on the interval $[0, T]$ with initial value u_0 if u satisfies

(1) $u \in C_w([0,T], H) \cap L^2([0,T], V)$, and $\partial_t u \in L^1([0,T], V')$, where V' is the dual space of V;

(2) the weak formulation of the NSE:

$$\begin{aligned}\int_\Omega (u\cdot\phi)(t)\,dxdydz - \int_\Omega (u\cdot\phi)(t_0)\,dxdydz \\ = \int_{t_0}^t \int_\Omega \left[u\cdot(\phi_t + \nu\Delta\phi)\right]\,dx\,ds \\ + \int_{t_0}^t \int_\Omega \left[(u\cdot\nabla)\phi\right]\cdot u\,dxdydz,\end{aligned}$$

for every $\phi \in \mathcal{V}$, and almost every t, $t_0 \in [0,T]$;

(3) the energy inequality:

$$\|u(t)\|_2^2 + \nu \int_{t_0}^t \|\nabla u(s)\|_2^2\,ds \le \|u(t_0)\|_2^2, \tag{6}$$

for every t and almost every t_0.

Moreover, if $u_0 \in V$, a weak solution is called strong solution of (1)–(3) on $[0,T]$ if, in addition, it satisfies

$$u \in C([0,T], V) \cap L^2([0,T], H^2(\Omega)), \text{ and } \partial_t u \in L^2([0,T], H).$$

In this case one also has energy equality in (6) instead of and inequality.

It is well known that the 2D Navier–Stokes equations have a unique weak and strong solutions which exist globally in time (cf., for example, [6], [16], [24], [27], [28]). In the $3D$ case, the weak solutions are known to exist globally in time. But, the uniqueness, regularity, and continuous dependence on initial data for weak solutions are still open problems.

Since Prodi [21], and Serrin in [23] provided a sufficient conditions for the global regularity of the $3D$ Navier–Stokes equations, many articles were dedicated for this subjects (for details see, for example, the survey papers [18], [29] and references therein). Most recently, there has been some progress along these lines (see, for example, [3], [8], [10], [11], [13], [25], [26], and references therein) which states, roughly speaking, that a strong solution u exists on the time interval $[0,T]$ for as long as

$$u \in L^p([0,T], L^q(\Omega)), \quad \text{with} \quad \frac{2}{p} + \frac{3}{q} \le 1, \quad \text{for} \quad q \ge 3, \tag{7}$$

or as long as

$$\nabla u \in L^s([0,T], L^r(\Omega)), \quad \text{with} \quad \frac{2}{s} + \frac{3}{r} \le 2, \quad \text{for} \quad r \ge 3/2. \tag{8}$$

In particular, in case that $r = 2$ and $s = 4$ in (8), namely,

$$\nabla u \in L^4([0,T], L^2(\Omega)), \tag{9}$$

then u is a strong solution on $[0,T]$.

Moreover, many sufficient regularity conditions were established in terms of only partial components of the velocity field, or partial components of gradient of velocity field of the $3D$ NSE (cf. e.g., [2], [4], [5], [12], [14], [15], [19], [20], [30]). In this paper, we provide improved sufficient conditions, in terms of only one direction derivative of the velocity field, that guarantee the global regularity of the $3D$ NSE. Specifically, if $u_0 \in V$, and if for some $T > 0$ we have

$$u_z \in L^\beta([0,T], L^\alpha(\Omega)); \quad \text{with } \alpha > \frac{27}{16}, \beta \geq 1, \text{and } \frac{3}{\alpha} + \frac{2}{\beta} \leq 2, \tag{10}$$

where u is a weak solution with the initial datum u_0, then u is a strong solution of the $3D$ Navier–Stokes equations which exists on the interval $[0,T]$. Notice that we have improved the condition in [15], which requires $\alpha \geq 9/4$. Moreover, our condition (10) implies that we only need (9) to be true on one direction derivative of the velocity field in order to get the regularity of the $3D$ NSE.

For convenience, we recall the following three-dimensional Sobolev and Ladyzhenskaya inequalities (see, e.g., [1], [6], [9], [17]). There exists a positive constant C_r such that

$$\begin{aligned}\|\psi\|_r &\leq C_r \|\psi\|_2^{\frac{6-r}{2r}} \left(\|\partial_x \psi\|_2 + \|\psi\|_2\right)^{\frac{r-2}{2r}} \left(\|\partial_y \psi\|_2 + \|\psi\|_2\right)^{\frac{r-2}{2r}} \left(\|\partial_z \psi\|_2 + \|\psi\|_2\right)^{\frac{r-2}{2r}} \\ &\leq C_r \|\psi\|_2^{\frac{6-r}{2r}} \|\psi\|_{H^1(\Omega)}^{\frac{3(r-2)}{2r}}, \end{aligned} \tag{11}$$

for every $\psi \in H^1(\Omega), 2 \leq r \leq 6$. And also,

$$\|\psi\|_r \leq C_r \left(\|\psi\|_2 + \|\psi\|_{3q}^{\frac{rq+6q}{(2+q)r}} \|\Delta\psi\|_2^{\frac{2r-6q}{(2+q)r}} \right), \tag{12}$$

for every $\psi \in H^2(\Omega), 3q \leq r$. More generally, one has

$$\|\phi\|_{L^{3q}} \leq C \left(\|\phi_z\|_q + \|\phi\|_q\right)^{\frac{1}{3}} \|\phi\|_{H^1}^{\frac{2}{3}}, \tag{13}$$

and

$$\|\phi\|_{L^{5q}} \leq C \|\phi^2\|_{H^1}^{\frac{2}{5}} \left(\|\phi_z\|_q + \|\phi\|_q\right)^{\frac{1}{5}}. \tag{14}$$

We have not found a reference to the above two inequalities (13) and (14). We present a proof here. It is clear that for every $q \geq 1$, we get

$$\begin{aligned}|\phi(x,y,x)|^{\frac{3q+2}{2}} &\leq C \left(\int_0^1 |\phi(x,y,z)|^{\frac{3q+2}{2}} \, dx + \int_0^1 |\phi(x,y,z)|^{\frac{3q}{2}} |\phi_x(x,y,z)| \, dx \right), \\ |\phi(x,y,x)|^{\frac{3q+2}{2}} &\leq C \left(\int_0^1 |\phi(x,y,z)|^{\frac{3q+2}{2}} \, dy + \int_0^1 |\phi(x,y,z)|^{\frac{3q}{2}} |\phi_y(x,y,z)| \, dy \right), \\ |\phi(x,y,x)|^{3q-2} &\leq C \left(\int_0^1 |\phi(x,y,z)|^{3q-2} \, dz + \int_0^1 |\phi(x,y,z)|^{3(q-1)} |\phi_z(x,y,z)| \, dz \right).\end{aligned}$$

As a result, we obtain

$$\begin{aligned}|\phi(x,y,z)|^{3q} \leq C &\left(\int_0^1 |\phi(x,y,z)|^{\frac{3q+2}{2}} \, dx + \int_0^1 |\phi(x,y,z)|^{\frac{3q}{2}} |\phi_x(x,y,z)| \, dx \right)^{\frac{1}{2}} \\ &\times \left(\int_0^1 |\phi(x,y,z)|^{\frac{3q+2}{2}} \, dy + \int_0^1 |\phi(x,y,z)|^{\frac{3q}{2}} |\phi_y(x,y,z)| \, dy \right)^{\frac{1}{2}} \\ &\times \left(\int_0^1 |\phi(x,y,z)|^{3q-2} \, dz + \int_0^1 |\phi(x,y,z)|^{3(q-1)} |\phi_z(x,y,z)| \, dz \right)^{\frac{1}{2}}.\end{aligned}$$

Thus, by Cauchy–Schwarz inequality, we get

$$\int_\Omega |\phi(x,y,z)|^{3q}\,dxdydz \le C\left(\int_\Omega |\phi(x,y,z)|^{\frac{3q+2}{2}}\,dxdydz + \int_\Omega |\phi|^{\frac{3q}{2}}\,|\phi_x|\,dxdydz\right)^{\frac12}$$
$$\times\left(\int_\Omega |\phi(x,y,z)|^{\frac{3q+2}{2}}\,dxdydz + \int_\Omega |\phi|^{\frac{3q}{2}}\,|\phi_y|\,dxdydz\right)^{\frac12}$$
$$\times\left(\int_\Omega |\phi(x,y,z)|^{3q-2}\,dxdydz + \int_\Omega |\phi|^{3(q-1)}\,|\phi_z|\,dxdydz\right)^{\frac12}$$
$$\le \|\phi\|_{3q}^{\frac{3q}{4}}\,\left(\|\phi\|_2+\|\phi_x\|_2\right)^{\frac12}\,\|\phi\|_{3q}^{\frac{3q}{4}}\,\left(\|\phi\|_2+\|\phi_y\|_2\right)^{\frac12}\,\|\phi\|_{3q}^{\frac{3(q-1))}{2}}\,\left(\|\phi\|_q+\|\phi_z\|_q\right)^{\frac12}.$$

Therefore,

$$\|\phi\|_{3q} \le C\,\|\phi_x\|_2^{\frac13}\,\|\phi_y\|_2^{\frac13}\,\|\phi_z\|_q^{\frac13},$$

and (13) follows. Similarly, for every $q \ge 1$, we get

$$|\phi(x,y,x)|^{\frac{5q+4}{2}} \le C\left(\int_0^1 |\phi(x,y,z)|^{\frac{5q+4}{2}}\,dx + \int_0^1 |\phi(x,y,z)|^{\frac{5q}{2}}\,|(\phi^2)_x(x,y,z)|\,dx\right),$$
$$|\phi(x,y,x)|^{\frac{5q+4}{2}} \le C\left(\int_0^1 |\phi(x,y,z)|^{\frac{5q+4}{2}}\,dy + \int_0^1 |\phi(x,y,z)|^{\frac{5q}{2}}\,|(\phi^2)_y(x,y,z)|\,dy\right),$$
$$|\phi(x,y,x)|^{5q-4} \le C\left(\int_0^1 |\phi(x,y,z)|^{5q-4}\,dz + \int_0^1 |\phi(x,y,z)|^{5(q-1)}\,|\phi_z(x,y,z)|\,dz\right),$$

As a result, we obtain

$$|\phi(x,y,z)|^{5q} \le C\,\left(\int_0^1 |\phi(x,y,z)|^{\frac{5q+4}{2}}\,dx + \int_0^1 |\phi(x,y,z)|^{\frac{5q}{2}}\,|(\phi^2)_x(x,y,z)|\,dx\right)^{\frac12}$$
$$\times\left(\int_0^1 |\phi(x,y,z)|^{\frac{5q+4}{2}}\,dy + \int_0^1 |\phi(x,y,z)|^{\frac{5q}{2}}\,|(\phi^2)_y(x,y,z)|\,dy\right)^{\frac12}$$
$$\times\left(\int_0^1 |\phi(x,y,z)|^{5q-4}\,dz + \int_0^1 |\phi(x,y,z)|^{5(q-1)}\,|\phi_z(x,y,z)|\,dz\right)^{\frac12}.$$

Thus, by Cauchy–Schwarz inequality, we get

$$\int_\Omega |\phi(x,y,z)|^{5q}\,dxdydz \le C\,\left(\int_\Omega |\phi|^{\frac{5q+4}{2}}\,dxdydz + \int_\Omega |\phi|^{\frac{5q}{2}}\,|(\phi^2)_x|\,dxdydz\right)^{\frac12}$$
$$\times\left(\int_\Omega |\phi|^{\frac{5q+4}{2}}\,dxdydz + \int_\Omega |\phi|^{\frac{5q}{2}}\,|(\phi^2)_y|\,dxdydz\right)^{\frac12}$$
$$\times\left(\int_\Omega |\phi|^{5q-4}\,dxdydz + \int_\Omega |\phi|^{5(q-1)}\,|\phi_z|\,dxdydz\right)^{\frac12}$$
$$\le \|\phi\|_{5q}^{\frac{5q}{4}}\,\left(\|\phi^2\|_2+\|(\phi^2)_x\|_2\right)^{\frac12}\,\|\phi\|_{5q}^{\frac{5q}{4}}\,\left(\|\phi^2\|_2+\|(\phi^2)_y\|_2\right)^{\frac12}$$
$$\times\|\phi\|_{5q}^{\frac{5(q-1)}{2}}\,\left(\|\phi\|_q+\|\phi_z\|_q\right)^{\frac12}.$$

Therefore,

$$\|\phi\|_{5q} \le C\,\left(\|\phi^2\|_2+\|(\phi^2)_x\|_2\right)^{\frac15}\,\left(\|\phi^2\|_2+\|(\phi^2)_y\|_2\right)^{\frac15}\,\left(\|\phi\|_q+\|\phi_z\|_2\right)^{\frac15},$$

and (14) follows.

2. **The main result.** In this section we will prove our main result, which states that the strong solution to system (1)–(3) exists on the interval $[0, T]$ provided the assumption (10) on u_z holds.

Theorem 2.1. *Let $u_0 \in V$, and let u be a Leray–Hopf weak solution to system (1)–(3) with the initial value u_0. Let $T > 0$, and suppose that u_z satisfies the condition (10), namely,*

$$\int_0^T \|u_z(s)\|_\alpha^\beta \, ds \le M, \quad \text{with } \alpha \ge \frac{27}{16}, 1 \le \beta < \infty, \text{ and } \frac{3}{\alpha} + \frac{2}{\beta} \le 2. \tag{15}$$

Then u is a strong solution of system (1)–(3) on the interval $[0, T]$. Moreover, it is the only weak solution on $[0, T]$ with the initial datum u_0.

Proof. It is clear that we only need to prove this theorem under $\frac{3}{\alpha} + \frac{2}{\beta} = 2$, i.e. $\beta = \frac{2\alpha}{2\alpha - 3}$. Also, in [15] the author has proved this theorem when $\alpha \ge 9/4$. Therefore, from now on, we only consider

$$27/16 \le \alpha \le 9/4, \quad \beta = \frac{2\alpha}{2\alpha - 3}. \tag{16}$$

By standard procedure for the $3D$ Navier–Stokes equations with periodic boundary condition (see, e.g., [6], [7], [18], [22], [24], [27], [28]) one can show that there exists a global in time Leray–Hopf weak solution to the system (1)–(3), if $u_0 \in H$. Furthermore, one can show the short time existence of a unique strong solution if $u_0 \in V$. In addition, this strong solution is the only weak solution, with the initial datum u_0, on the maximal interval of existence of the strong solution.

Suppose that u is the strong solution with initial value $u_0 \in V$ such that $u \in C([0, \mathcal{T}^*), V) \cap L^2([0, \mathcal{T}^*), H^2(\Omega))$, where $[0, \mathcal{T}^*)$ is the maximal interval of existence of the unique strong solution. If $\mathcal{T}^* \ge T$ then there is nothing to prove. If, on the other hand, $\mathcal{T}^* < T$ our strategy is to show that the H^1 norm of this strong solution is bounded on $[0, \mathcal{T}^*)$, provided condition (10) is valid. As a result $[0, \mathcal{T}^*)$ cannot be a maximal interval of existence and consequently $\mathcal{T}^* \ge T$. Which will conclude our proof.

From now on we focus on the strong solution, u, on its maximal interval of existence $[0, \mathcal{T}^*)$, where we assume that $\mathcal{T}^* < T$. As we have observed earlier the strong solution u will also be the only weak solution on the interval $[0, \mathcal{T}^*)$. Therefore, by the energy inequality (6) for Leray–Hopf weak solutions we have (see, for example, [6], [27] or [28] for details)

$$\|u(t)\|_2^2 + \nu \int_0^t \|\nabla u(s)\|_2^2 \, ds \le K_1, \tag{17}$$

where

$$K_1 = \|u_0\|_2^2. \tag{18}$$

Moreover, by (16) and (17), we get

$$\int_0^t \|u(s)\|_\alpha^\beta \, ds \le \begin{cases} \int_0^t \|u(s)\|_2^\beta \, ds & 27/16 \le \alpha \le 2 \\ \int_0^t \|u(s)\|_2^{\frac{6-\alpha}{2\alpha-3}} \|u(s)\|_{H^1}^{\frac{3(\alpha-2)}{2\alpha-3}} \, ds & 2 \le \alpha \le 9/4 \end{cases} \tag{19}$$

$$\le C K_1^{\beta/2}, \tag{20}$$

For convenience , from now on, we will rewrite the Navier–Stokes equations as

$$\frac{\partial v}{\partial t} - \nu \Delta_h v - \nu v_{zz} + (v \cdot \nabla_h) v + w v_z + \nabla_h p = 0, \tag{21}$$

$$\frac{\partial w}{\partial t} - \nu \Delta_h w - \nu w_{zz} + v \cdot \nabla_h w + w w_z + p_z = 0, \tag{22}$$

$$\nabla_h \cdot v + w_z = 0, \tag{23}$$

where $v = (v_1, v_2)$ is the horizontal velocity field components and w is the vertical velocity component. Namely, $u = (v, w)$. We set $\nabla_h = (\partial_x, \partial_y)$ to be the horizontal gradient operator and $\Delta_h = \partial_x^2 + \partial_y^2$ the horizontal Laplacian, respectively. Next, let us show that the H^1 norm of the strong solution u is bounded on interval $[0, \mathcal{T}^*)$. Denote by

$$X(t) = \|u(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \|u_z(t)\|_2^2 + \|w(t)\|_4^4, \tag{24}$$

$$Y(t) = \int_0^t \left(\|\Delta v(s)\|_2^2 + \|\nabla u_z(s)\|_2^2 + \|\nabla(w^2)(s)\|_2^2 \right) ds, \tag{25}$$

$$V(t) = \int_0^t \left(1 + \|u(s)\|_2^2 + \|\nabla u(s)\|_2^2 + (\|u\|_\alpha + \|u_z(s)\|_\alpha)^{\frac{2\alpha}{2\alpha-3}} \right) \times \left(\|\nabla v(s)\|_2^2 + \|u_z(s)\|_2^2 + \|w^2(s)\|_2^2 \right) ds, \tag{26}$$

where α is as in (10).

2.1. L^4**estimate.** Taking the inner product of the equation (22) with w^3 in $L^2(\Omega)$, we get

$$\frac{1}{4} \frac{d\|w^2\|_2^2}{dt} + \frac{3\nu}{4} \|\nabla(w^2)\|_2^2 = -\int_\Omega (u \cdot \nabla w)\, w^3 \, dxdydz - \int_\Omega p_z w^3 \, dxdydz. \tag{27}$$

By integration by parts, and the incompressibility condition (2) we get

$$\int_\Omega (u \cdot \nabla w)\, w^3 \, dxdydz = 0. \tag{28}$$

By the standard procedure one can get

$$-\Delta p_z = 2\nabla \cdot (\nabla \cdot (u\, u_z)).$$

Let us denote by

$$p_1 = 2(-\Delta)^{-1} \left[\nabla \cdot (\nabla \cdot (v\, u_z)) \right], \tag{29}$$

and

$$p_2 = 2(-\Delta)^{-1} \left[\nabla \cdot (\nabla \cdot (w\, u_z)) \right]. \tag{30}$$

Moreover, by Hölder's inequality and (11), we obtain

$$\begin{aligned} \left| \int_\Omega p_z w^3 \, dxdydz \right| &\le \left| \int_\Omega p_1 w^3 \, dxdydz \right| + \left| \int_\Omega p_2 w^3 \, dxdydz \right| \\ &\le C \|p_1\|_{\frac{6\alpha(3\alpha-1)}{2\alpha^2+23\alpha-18}} \|w^3\|_{\frac{6\alpha(3\alpha-1)}{16\alpha^2-29\alpha+18}} + \|p_2\|_{\frac{4\alpha}{\alpha+3}} \|w^3\|_{\frac{4\alpha}{3(\alpha-1)}} \\ &\le C \|v\, u_z\|_{\frac{6\alpha(3\alpha-1)}{2\alpha^2+23\alpha-18}} \|w^3\|_{\frac{6\alpha(3\alpha-1)}{16\alpha^2-29\alpha+18}} + \|w\, u_z\|_{\frac{4\alpha}{\alpha+3}} \|w^3\|_{\frac{4\alpha}{3(\alpha-1)}} \\ &\le C \|v\|_{\frac{6\alpha(3\alpha-1)}{(\alpha+4)(2\alpha-3)}} \|u_z\|_\alpha \|w^3\|_{\frac{6\alpha(3\alpha-1)}{16\alpha^2-29\alpha+18}} + \|w\|_{\frac{4\alpha}{\alpha-1}} \|u_z\|_\alpha \|w^3\|_{\frac{4\alpha}{3(\alpha-1)}}. \end{aligned} \tag{31}$$

Let us estimate the above term by term. First, by (11), we get

$$\begin{aligned}
&\|w\|_{\frac{4\alpha}{\alpha-1}}\|u_z\|_\alpha \ \|w^3\|_{\frac{4\alpha}{3(\alpha-1)}} = \|w\|^4_{\frac{4\alpha}{\alpha-1}}\|u_z\|_\alpha \\
&= \|w^2\|^2_{\frac{2\alpha}{\alpha-1}}\|u_z\|_\alpha \\
&\le C\left(\|w^2\|_2^2 + \|w^2\|_2^{\frac{2\alpha-3}{\alpha}}\|\nabla w^2\|_2^{\frac{3}{\alpha}}\right)\|u_z\|_\alpha.
\end{aligned}$$

Next, applying (12) with $r = \frac{6\alpha(3\alpha-1)}{(\alpha+4)(2\alpha-3)}$ and $q=\alpha$, we obtain

$$\|v\|_{\frac{6\alpha(3\alpha-1)}{(\alpha+4)(2\alpha-3)}} \le C\left(\|v\|_2 + \|v\|_{3\alpha}^{\frac{5\alpha-6}{3\alpha-1}}\|\Delta v\|_2^{\frac{5-2\alpha}{3\alpha-1}}\right).$$

By interpolation inequality we get

$$\|w^3\|_{\frac{6\alpha(3\alpha-1)}{16\alpha^2-29\alpha+18}} = \|w\|^3_{\frac{18\alpha(3\alpha-1)}{16\alpha^2-29\alpha+18}} \le C\|w\|_4^{\frac{2(16\alpha-27)}{3(3\alpha-1)}}\|w\|_{5\alpha}^{\frac{5(9-\alpha)}{3(3\alpha-1)}}.$$

By (13), we reach

$$\|v\|_{3\alpha}^{\frac{5\alpha-6}{3\alpha-1}} \le C\left(\|v\|_2 + \|\nabla v\|_2\right)^{\frac{2(5\alpha-6)}{3(3\alpha-1)}}\left(\|v\|_\alpha + \|v_z\|_\alpha\right)^{\frac{5\alpha-6}{3(3\alpha-1)}},$$

and by (14), we obtain

$$\|w\|_{5\alpha}^{\frac{5(9-\alpha)}{3(3\alpha-1)}} \le C\left(\|w^2\|_2 + \|\nabla w^2\|_2\right)^{\frac{2(9-\alpha)}{3(3\alpha-1)}}\left(\|w\|_\alpha + \|w_z\|_\alpha\right)^{\frac{9-\alpha}{3(3\alpha-1)}}.$$

Thus,

$$\begin{aligned}
&\frac{1}{4}\frac{d\|w^2\|_2^2}{dt} + \frac{3\nu}{4}\|\nabla(w^2)\|_2^2 \\
&\le C\left(\|v\|_2 + (\|v\|_2+\|\nabla v\|_2)^{\frac{2(5\alpha-6)}{3(3\alpha-1)}}(\|v\|_\alpha+\|v_z\|_\alpha)^{\frac{5\alpha-6}{3(3\alpha-1)}}\|\Delta v\|_2^{\frac{5-2\alpha}{3\alpha-1}}\right) \\
&\quad\times\|u_z\|_\alpha\ \|w^2\|_2^{\frac{16\alpha-27}{3(3\alpha-1)}}\left(\|w^2\|_2+\|\nabla w^2\|_2\right)^{\frac{2(9-\alpha)}{3(3\alpha-1)}}(\|w\|_\alpha+\|w_z\|_\alpha)^{\frac{9-\alpha}{3(3\alpha-1)}} \\
&\quad+C\left(\|w^2\|_2^2 + \|w^2\|_2^{\frac{2\alpha-3}{\alpha}}\|\nabla w^2\|_2^{\frac{3}{\alpha}}\right)\|u_z\|_\alpha \\
&\le C\Bigg(\|v\|_2(\|u\|_\alpha+\|u_z\|_\alpha)^{\frac{6+8\alpha}{3(3\alpha-1)}}\ \|w^2\|_2^{\frac{14\alpha-9}{3(3\alpha-1)}} \\
&\quad+\|v\|_2(\|u\|_\alpha+\|u_z\|_\alpha)^{\frac{6+8\alpha}{3(3\alpha-1)}}\ \|w^2\|_2^{\frac{16\alpha-27}{3(3\alpha-1)}}\ \|\nabla w^2\|_2^{\frac{2(9-\alpha)}{3(3\alpha-1)}} \\
&\quad+\|v\|_{H^1}^{\frac{2(5\alpha-6)}{3(3\alpha-1)}}(\|u\|_\alpha+\|u_z\|_\alpha)^{\frac{13\alpha}{3(3\alpha-1)}}\|\Delta v\|_2^{\frac{5-2\alpha}{3\alpha-1}}\ \|w^2\|_2^{\frac{14\alpha-9}{3(3\alpha-1)}} \\
&\quad+\|v\|_{H^1}^{\frac{2(5\alpha-6)}{3(3\alpha-1)}}(\|u\|_\alpha+\|u_z\|_\alpha)^{\frac{13\alpha}{3(3\alpha-1)}}\|\Delta v\|_2^{\frac{5-2\alpha}{3\alpha-1}}\ \|w^2\|_2^{\frac{16\alpha-27}{3(3\alpha-1)}}\ \|\nabla w^2\|_2^{\frac{2(9-\alpha)}{3(3\alpha-1)}}\Bigg) \\
&\quad+C\left(\|w^2\|_2^2\|u_z\|_\alpha + \|w^2\|_2^{\frac{2\alpha-3}{\alpha}}\|\nabla w^2\|_2^{\frac{3}{\alpha}}\ \|u_z\|_\alpha\right).
\end{aligned}$$

Thanks to Gronwall's inequality and (17), (15) and notations (24)–(26), we obtain

$$\begin{aligned}
&\|w^2(t)\|_2^2 + \nu\int_0^t\|\nabla(w^2)(s)\|_2^2\,ds \qquad (32)\\
&\le \|w^2(t=0)\|_2^2 + C\left(1 + V(t) + V(T)^{\frac{26\alpha-39}{6(3\alpha-1)}}\ Y(t)^{\frac{33-8\alpha}{6(3\alpha-1)}} + V(t)^{\frac{2\alpha-3}{2\alpha}}Y(t)^{\frac{3}{2\alpha}}\right).
\end{aligned}$$

2.2. H^1 **estimates.** Taking the inner product of the equation (21) with $-\Delta v$ and the equation (1) with $-u_{zz}$ in L^2, and using the fact that the Stokes operator is the same as the Laplacian operator under periodic boundary conditions, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d(\|\nabla v\|_2^2+\|u_z\|_2^2)}{dt}+\nu(\|\Delta v\|_2^2+\|\nabla u_z\|)\\&=\int_\Omega [(u\cdot\nabla)v]\cdot\Delta v\;dxdydz+\int_\Omega \nabla_h p\Delta v\;dxdydz+\int_\Omega [(u\cdot\nabla)u]\cdot u_{zz}\;dxdydz\\&=\int_\Omega \left[(v\cdot\nabla_h)v\cdot\Delta_h v+(v\cdot\nabla_h)v\cdot v_{zz}+wv_z\cdot\Delta v\right]\;dxdydz+\int_\Omega \nabla_h p\Delta v\;dxdydz\\&\quad+\int_\Omega [(u\cdot\nabla)u]\cdot u_{zz}\;dxdydz.\end{aligned}$$

By integration by parts, we get

$$\int_\Omega (v\cdot\nabla_h)v\cdot\Delta_h v\;dxdydz=\int_\Omega w_z\left[|\nabla_h v|^2+\partial_x v_2\,\partial_y v_1-\partial_x v_1\,\partial_y v_2\right]\;dxdydz \tag{33}$$

By integration by parts in the above we have

$$\int_\Omega((v\cdot\nabla_h)v+wv_z)\cdot v_{zz}\;dxdydz=-\int_\Omega((v_z\cdot\nabla_h)v+w_z\,v_z)\cdot v_z\;dxdydz, \tag{34}$$

and

$$\begin{aligned}&\int_\Omega \nabla_h p\Delta v\;dxdydz=-\int_\Omega \Delta p\;(\nabla_h\cdot v)\;dxdydz\\&=-\int_\Omega (2(\partial_x v_1)^2+2(\partial_y v_2)^2+2\partial_x v_2\,\partial_y v_1+2\partial_x v_1\,\partial_y v_2+2\nabla_h w\cdot v_z)\;w_z\;dxdydz\\&=-\int_\Omega (2(\partial_x v_1)^2+2(\partial_y v_2)^2+2\partial_x v_2\,\partial_y v_1+2\partial_x v_1\,\partial_y v_2)\;w_z\;dxdydz\\&\quad+\int_\Omega 2w(\nabla_h v_z\;w_z+v_z\,\nabla_h w_z)\;dxdydz.\end{aligned} \tag{35}$$

As a result of all of the above, and Hölder inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d(\|\nabla v\|_2^2+\|u_z\|_2^2)}{dt}+\nu(\|\Delta v\|_2^2+\|\nabla u_z\|)\\&\le C\int_\Omega \left[|u_z|\,|\nabla v|^2+|w|\,|u_z|\,|\nabla^2 v|+|w|\,|u_z|\,|\nabla u_z|\right]\;dxdydz\\&\le C\left[\|u_z\|_\alpha\|\nabla v\|^2_{\frac{2\alpha}{\alpha-1}}+\|w\|_{\frac{2\alpha(7\alpha-3)}{6\alpha^2+6-11\alpha}}\|u_z\|_{\frac{2\alpha(7\alpha-3)}{\alpha^2+8\alpha-6}}(\|\Delta v\|_2+\|\nabla u_z\|_2)\right].\end{aligned}$$

By (11)–(14), we obtain

$$\begin{aligned}&\|\nabla v\|^2_{\frac{2\alpha}{\alpha-1}}\le C\left[\|\nabla v\|_2^2+\|\nabla v\|_2^{\frac{2\alpha-3}{\alpha}}\|\Delta v\|_2^{\frac{3}{\alpha}}\right],\\&\|w\|_{\frac{2\alpha(7\alpha-3)}{6\alpha^2+6-11\alpha}}\le C\|w\|_4^{\frac{6(2\alpha-3)}{7\alpha-3}}\|w\|_{5\alpha}^{\frac{5(3-\alpha)}{7\alpha-3}}\\&\qquad\le C\|w^2\|_2^{\frac{3(2\alpha-3)}{7\alpha-3}}\left(\|w^2\|_2^{\frac{2(3-\alpha)}{7\alpha-3}}+\|\nabla w^2\|_2^{\frac{2(3-\alpha)}{7\alpha-3}}\right)\;(\|w\|_\alpha+\|w_z\|_\alpha)^{\frac{3-\alpha}{7\alpha-3}},\\&\|u_z\|_{\frac{2\alpha(7\alpha-3)}{\alpha^2+8\alpha-6}}\le C\left(\|u_z\|_2+\|u_z\|_\alpha^{\frac{4\alpha-3}{7\alpha-3}}\|\nabla u_z\|_2^{\frac{3\alpha}{7\alpha-3}}\right).\end{aligned}$$

Therefore, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d(\|\nabla v\|_2^2+\|u_z\|_2^2)}{dt}+\nu(\|\Delta v\|_2^2+\|\nabla u_z\|) \qquad (36)\\
&\leq C\left[\|\nabla v\|_2^2+\|\nabla v\|_2^{\frac{2\alpha-3}{\alpha}}\|\Delta v\|_2^{\frac{3}{\alpha}}\right]\|u_z\|_\alpha\\
&\quad +C\|w^2\|_2^{\frac{3(2\alpha-3)}{7\alpha-3}}\left(\|w^2\|_2^{\frac{2(3-\alpha)}{7\alpha-3}}+\|\nabla w^2\|_2^{\frac{2(3-\alpha)}{7\alpha-3}}\right)\left(\|w\|_\alpha+\|w_z\|_\alpha\right)^{\frac{3-\alpha}{7\alpha-3}}\\
&\quad\quad\times\left(\|u_z\|_2+\|u_z\|_\alpha^{\frac{4\alpha-3}{7\alpha-3}}\|\nabla u_z\|_2^{\frac{3\alpha}{7\alpha-3}}\right)(\|\Delta v\|_2+\|\nabla u_z\|_2)\\
&\leq C\left[\|\nabla v\|_2^2\|u_z\|_\alpha+\|\nabla v\|_2^{\frac{2\alpha-3}{\alpha}}\|\Delta v\|_2^{\frac{3}{\alpha}}\|u_z\|_\alpha\right]+ \qquad (37)\\
&\quad +C\left(\|\Delta v\|_2+\|\nabla u_z\|_2\right)\Bigg[\|u_z\|_2\left(\|u\|_\alpha+\|u_z\|_\alpha\right)^{\frac{3-\alpha}{7\alpha-3}}\|w^2\|_2^{\frac{4\alpha}{7\alpha-3}}\\
&\quad\quad +\left(\|u\|_\alpha+\|u_z\|_\alpha\right)^{\frac{3\alpha}{7\alpha-3}}\|\nabla u_z\|_2^{\frac{3\alpha}{7\alpha-3}}\|w^2\|_2^{\frac{4\alpha}{7\alpha-3}}\\
&\quad\quad +\|u_z\|_2\left(\|u\|_\alpha+\|u_z\|_\alpha\right)^{\frac{3-\alpha}{7\alpha-3}}\|w^2\|_2^{\frac{3(2\alpha-3)}{7\alpha-3}}\|\nabla w^2\|_2^{\frac{2(3-\alpha)}{7\alpha-3}}\\
&\quad\quad +\left(\|u\|_\alpha+\|u_z\|_\alpha\right)^{\frac{3\alpha}{7\alpha-3}}\|\nabla u_z\|_2^{\frac{3\alpha}{7\alpha-3}}\|w^2\|_2^{\frac{3(2\alpha-3)}{7\alpha-3}}\|\nabla w^2\|_2^{\frac{2(3-\alpha)}{7\alpha-3}}\Bigg].
\end{aligned}
$$

Thanks to Gronwall's inequality and (17), we obtain

$$
\begin{aligned}
&\|\nabla v(t)\|_2^2+\|u_z(t)\|_2^2+\nu\int_0^t\left(\|\Delta v(s)\|_2^2+\|\nabla u_z(s)\|\right)ds \qquad (38)\\
&\leq\|\nabla u(t=0)\|_2^2+C\left(1+V(t)+V(t)^{\frac{2\alpha-3}{2\alpha}}Y(t)^{\frac{3}{2\alpha}}+V(t)^{\frac{3(2\alpha-3)}{2(7\alpha-3)}}Y(t)^{\frac{3+8\alpha}{2(7\alpha-3)}}\right),
\end{aligned}
$$

where $X(t), Y(t)$ and $V(t)$ are as in (24)–(26), respectively. Therefore, by (32) and (38), we reach

$$
\begin{aligned}
&X(t)+Y(t)\leq\|u(t=0)\|_{H^1}^2+\|w^2(t=0)\|_2^2+C(1+V(t)) \qquad (39)\\
&+C\left(V(t)^{\frac{26\alpha-39}{6(3\alpha-1)}}Y(t)^{\frac{33-8\alpha}{6(3\alpha-1)}}+V(t)^{\frac{2\alpha-3}{2\alpha}}Y(t)^{\frac{3}{2\alpha}}+V(t)^{\frac{3(2\alpha-3)}{2(7\alpha-3)}}Y(t)^{\frac{3+8\alpha}{2(7\alpha-3)}}\right).
\end{aligned}
$$

Thanks to Young's inequality, we get

$$
X(t)+Y(t)\leq\|\nabla u_0\|_2^2+\|u_0\|_4^4+C(K_1,\alpha,t)+CV(t), \qquad (40)
$$

where $C(K_1,\alpha,t)$ is a constant depended on K_1,α,t. Therefore, by Gronwall's inequality we obtain

$$
X(t)+Y(t)\leq C\left[\|\nabla u_0\|_2^2+\|u_0\|_4^4+C(K_1,\alpha,t)\right]\, e^{C(1+K_1+M)}, \qquad (41)
$$

for all $t\in[0,\mathcal{T}^*)$, where M is as in (15). Therefore, the $\|\nabla v\|_2$ and $\|u_z\|_2$ are for all $t\in[0,\mathcal{T}^*)$. Following the previous results (e.g., [4], [14]), we have that the H^1 norm of the strong solution u is bounded on the maximal interval of existence $[0,\mathcal{T}^*)$, which leads to a contradiction. This completes the proof of Theorem 2.1. □

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