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# A 'horizontal' hyper-diffusion three-dimensional thermocline planetary geostrophic model: well-posedness and long-time behaviour

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**Abstract**
In this paper, we study a three-dimensional thermocline planetary geostrophic 'horizontal' hyper-diffusion model of the gyre-scale midlatitude ocean. We show the global existence and uniqueness of the weak and strong solutions to this model. Moreover, we establish the existence of a finite-dimensional global attractor to this dissipative evolution system. Preliminary computational tests indicate that our hyper-diffusion model does not exhibit any of the non-physical instabilities near the lateral boundary which are observed numerically in other models.

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## 1. Introduction

Let $\Omega = M \times (-h, 0) \in \mathbb{R}^3$, where $M$ is a smooth domain in $\mathbb{R}^2$, or $M = (0, 1) \times (0, 1)$. The planetary geostrophic (PG) equations with friction and diffusion can be written as:

$$\nabla p + f\vec{k} \times v + \mathcal{D} = 0, \quad (1)$$

$$\partial_z p + T = 0, \quad (2)$$

$$\nabla \cdot v + \partial_z w = 0, \tag{3}$$

$$\partial_t T + v \cdot \nabla T + w \partial_z T = K_v T_{zz} - \nabla \cdot q(T), \tag{4}$$

where $\mathcal{D}$ is the friction or dissipation of momentum and $\nabla \cdot q(T) - K_v T_{zz}$ is the heat diffusion.

Naturally, there are two friction schemes: one is the linear drag, i.e. $\mathcal{D} = \epsilon v$; the other is the conventional eddy viscosity, i.e. $\mathcal{D} = -\epsilon(A_h \Delta v + A_v \partial_{zz} v)$, where $\Delta = \Delta_h = \partial_x^2 + \partial_y^2$, and $A_h$ and $A_v$ are small positive constants. The PG model with conventional eddy viscosity and the diffusion $q(T) = -K_h \nabla T$, has been studied in [2, 21, 22]. In this paper, we are concerned with the case of linear drag. As explained below, in a closed basin and with a linear drag scheme, it is appropriate that $\nabla \cdot q$ be a fourth-order diffusion.

Under appropriate physical boundary conditions, which we derive rigorously, we prove, in this paper, the global existence and uniqueness of the weak and strong solutions to this model. Moreover, we establish the existence of a finite-dimensional global attractor to this dissipative evolution system.

### *1.1. Previous results and physical discussion*

The PG models, the adiabatic and inviscid form of 'thermocline' or 'Phillips type II' equations, are derived by standard scaling analysis for gyre-scale oceanic motion (see [13, 15, 16, 28, 29]). In their simplest dimensionless $\beta$-plane form, these equations are:

$$\partial_x p - f v_2 = 0, \tag{5}$$

$$\partial_y p + f v_1 = 0, \tag{6}$$

$$\partial_z p + T = 0, \tag{7}$$

$$\nabla \cdot v + \partial_z w = 0, \tag{8}$$

$$\partial_t T + v \cdot \nabla T + w \partial_z T = 0, \tag{9}$$

in the domain $\Omega = \{(x, y, z) : (x, y) \in M \subset \mathbb{R}^2, \text{ and } z \in (-h, 0)\}$. Here $\nabla = (\partial_x, \partial_y)$, $v = (v_1, v_2)$ denotes the horizontal velocity field, $w$ the vertical velocity, $p$ is the pressure, $T$ is the temperature and $f = f_0 + \beta y$ is the Coriolis parameter. A diffusive term, $\kappa_v \partial_z^2 T$, is commonly added to equation (9) as a leading order approximation to the effect of microscale turbulent mixing. Thus equation (9) becomes:

$$\partial_t T + v \cdot \nabla T + w \partial_z T = \kappa_v \partial_z^2 T. \tag{10}$$

In [24] it is argued, based on physical grounds, that in a closed ocean basin and with the no-normal-flow boundary conditions, this model can be solved only in restricted domains which are bounded away from the lateral boundaries, $\partial M \times (-h, 0)$. Thus, it cannot be utilized in the study of the large-scale circulation. Furthermore, it has been pointed out, numerically, by [4] that arbitrarily short linear disturbances (disturbances that are supported at small spatial scales) will grow arbitrarily fast when the flow becomes baroclinically unstable. This non-physical growth at small scales is a signature of mathematical ill-posedness of this model near unstable baroclinic modes. Therefore, Samelson and Vallis proposed in [24] various simple friction and diffusion schemes to overcome these physical and numerical difficulties. In particular, they propose a linear drag (or viscosity) in the horizontal momentum equations and a horizontal diffusion in the thermodynamic equation (10). Thus, the full dimensionless system becomes:

$$\partial_x p - f v_2 = -\epsilon v_1 + \nu_v \partial_z^2 v_1 + \nu_h \Delta_h v_1, \tag{11}$$

$$\partial_y p + f v_1 = -\epsilon v_2 + \nu_v \partial_z^2 v_2 + \nu_h \Delta_h v_2, \tag{12}$$

$$\partial_z p + T = 0, \tag{13}$$

$$\nabla v + \partial_z w = 0, \tag{14}$$

$$\partial_t T + v \cdot \nabla T + w\partial_z T = \kappa_v \partial_z^2 T + \kappa_h \Delta_h T, \tag{15}$$

where $\Delta_h = (\partial^2/\partial x^2) + (\partial^2/\partial y^2)$. Note that the incompressibility and hydrostatic balance are retained in the above system. Here, the coefficients $\nu_h$, $\nu_v$, $\epsilon$ and $\kappa_h$ are non-negative and small.

In the case where $\nu_h > 0$, $\nu_v > 0$, $\epsilon \geqslant 0$, $\kappa_v > 0$ and $\kappa_h > 0$, with the ocean being driven by the wind stresses at the top surface, and with no-slip boundary conditions and no-heat fluxes on the side walls, $\Gamma_s = \partial M \times (-h, 0)$, and at the bottom, $M \times \{z = -h\}$, the above system has been studied analytically in [21, 22]. Under this type of boundary conditions the first two authors have improved in [2] the results of [21, 22] and established global regularity and well-posedness to the system (11)–(15). In particular, they show in [2] the existence of a finite-dimensional global attractor to this viscous three-dimensional PG model. This global well-posedness result provides a rigorous justification to the scheme suggested by Samelson and Vallis [24] under the above conditions on the coefficients.

On the other hand, the case where $\nu_h = 0$, $\nu_v = 0$, $\epsilon > 0$, $\kappa_v > 0$ and $\kappa_h > 0$, has been suggested in related schemes by other authors [8, 17–19, 30, 31]. Equations (11)–(15) become

$$\partial_x p - f v_2 = -\epsilon v_1, \tag{16}$$

$$\partial_y p + f v_1 = -\epsilon v_2, \tag{17}$$

$$\partial_z p + T = 0, \tag{18}$$

$$\nabla v + \partial_z w = 0, \tag{19}$$

$$\partial_t T + v \cdot \nabla T + w\partial_z T = \kappa_v \partial_z^2 T + \kappa_h \Delta_h T. \tag{20}$$

In this situation, and following [24], the frictional-geostrophic relations (16) and (17) can be solved locally for the horizontal velocities to give:

$$v_1 = -\gamma(\epsilon p_x + f p_y), \qquad v_2 = \gamma(f p_x - \epsilon p_y), \tag{21}$$

where $\gamma = (f^2 + \epsilon^2)^{-1}$. Assuming we have a nice solution up to the boundary, equations (21) and (18) imply:

$$\partial_z v_1 = -\gamma(\epsilon T_x + f T_y), \qquad \partial_z v_2 = \gamma(f T_x - \epsilon T_y). \tag{22}$$

As a result of the above, the no-normal-flow boundary condition on the lateral boundary yields the following boundary condition on the temperature:

$$\epsilon \frac{\partial T}{\partial n} + f \frac{\partial T}{\partial s} = 0, \qquad \text{on the lateral boundary, } \Gamma_s = \partial M \times (-h, 0), \tag{23}$$

where $\partial T/\partial n$ and $\partial T/\partial s$ denote the normal and right-hand tangential derivative, respectively, so that with $\vec{e} = (\epsilon n_1 - f n_2, f n_1 + \epsilon n_2)/(\epsilon^2 + f^2)^{1/2}$, we have the following oblique boundary condition on the temperature.

$$\frac{\partial T}{\partial \vec{e}} = 0, \qquad \text{on } \Gamma_s. \tag{24}$$

On the other hand, the no-heat flux at the lateral boundary gives:

$$-\kappa_h \frac{\partial T}{\partial n} = 0, \qquad \text{on the lateral boundary, } \partial M \times (-h, 0). \tag{25}$$

Based on the above it is argued in [24] that in the presence of rotation, i.e. $f \neq 0$, and in order to be able to satisfy both boundary conditions (23) and (25) one has to add to the thermodynamics equation a higher order (biharmonic) horizontal diffusion. Otherwise, the problem (16)–(20) subject to the additional boundary conditions (23) and (25) is over determined and, hence, ill-posed, which explains the cause for the observed numerical instabilities near the lateral boundary. In order to resolve this discrepancy Samelson and Vallis propose in [24] the following 'horizontal hyper-diffusion' PG model:

$$\partial_x p - f v_2 = -\epsilon v_1, \tag{26}$$

$$\partial_y p + f v_1 = -\epsilon v_2, \tag{27}$$

$$\partial_z p + T = 0, \tag{28}$$

$$\nabla \cdot v + \partial_z w = 0, \tag{29}$$

$$\partial_t T + v \cdot \nabla T + w \partial_z T = \kappa_v \partial_z^2 T + \kappa_h \Delta_h T - \lambda \Delta_h^2 T, \tag{30}$$

subject to the no-normal-flow together with the boundary conditions given in (23) on the lateral boundary. However, the no-heat-flux boundary condition (25) is replaced by the boundary condition:

$$-\kappa_h \frac{\partial T}{\partial n} + \lambda \Delta_h \frac{\partial T}{\partial n} = 0, \qquad \text{on the lateral boundary, } \Gamma_s = \partial M \times (-h, 0). \tag{31}$$

It is worth stressing that the differences between the boundary conditions (23) and (25) is due to the Coriolis parameter. Therefore, it is natural to require $\lambda \to 0$ when $f \to 0$. Since the diffusion operator $\kappa_v \partial_z^2 T + \kappa_h \Delta_h T - \lambda \Delta_h^2 T$ in the model (26)–(30) with the boundary conditions (24) and (31) may not positive, the numerical instabilities have been observed near the lateral boundary.

Motivated by [24] we propose below a variant of the system (34)–(43) which is globally well-posed and which possesses a finite-dimensional global attractor.

### *1.2. The PG model and boundary conditions*

Our goal in this paper is to study the model with $\mathcal{D} = \epsilon v$, and with 'horizontal' hyper-diffusion

$$q(T) = \lambda H \nabla(\nabla \cdot (H^T \nabla T)) - K_h \nabla T + \mu \nabla T_{zz}, \tag{32}$$

where

$$H = \begin{pmatrix} 1 & -\dfrac{f}{\epsilon} \\ -\dfrac{f}{\epsilon} & 1 \end{pmatrix} \tag{33}$$

and $H^T$ is the transpose of $H$, and $\lambda$, $\mu$ and $\epsilon$ are small positive constants. That is, we study the following model:

$$\nabla p + f \vec{k} \times v + \epsilon v = 0, \tag{34}$$

$$\partial_z p + T = 0, \tag{35}$$

$$\nabla \cdot v + \partial_z w = 0, \tag{36}$$

$$\partial_t T + v \cdot \nabla T + w \partial_z T = Q + K_v T_{zz} - \nabla \cdot (\lambda H \nabla(\nabla \cdot (H^T \nabla T)) - K_h \nabla T + \mu \nabla T_{zz}), \tag{37}$$

where $Q$ is a given heat source.

Now we impose the appropriate boundary conditions to this friction and hyper-diffusion PG model (34)–(37). The natural boundary conditions are the no-normal-flow condition for the velocity field $(v, w)$ and the non-flux boundary condition for the temperature $T$

(see, e.g. [14, 23–25]):

$$\text{on } \Gamma_u : w = 0, \tag{38}$$

$$\text{on } \Gamma_b : w = 0, \tag{39}$$

$$\text{on } \Gamma_s : v \cdot \vec{n} = 0, \tag{40}$$

$$\text{on } \Gamma_u : -K_v \frac{\partial T}{\partial z} = \alpha(T - T^*), \tag{41}$$

$$\text{on } \Gamma_b : \frac{\partial T}{\partial z} = 0, \tag{42}$$

$$\text{on } \Gamma_s : q(T) \cdot \vec{n} = 0, \tag{43}$$

where $\Gamma_u$, $\Gamma_b$ and $\Gamma_s$ denote the boundary of $\Omega$ defined as:

$$\Gamma_u = \{(x, y, z) \in \Omega : z = 0\}, \tag{44}$$

$$\Gamma_b = \{(x, y, z) \in \Omega : z = -h\}, \tag{45}$$

$$\Gamma_s = \{(x, y, z) \in \Omega : (x, y) \in \partial M\}, \tag{46}$$

$\alpha$ is a positive constant, $\vec{n} = (n_1, n_2)$ is the normal vector of $\Gamma_s$, and $T^*(x, y)$ is a typical top surface temperature profile. Furthermore, the no-normal-flow condition implies a boundary condition on the temperature, as explained in (21)–(23):

$$\frac{\partial T}{\partial \vec{e}} = 0, \qquad \text{on } \Gamma_s, \tag{47}$$

where $\vec{e} = (\epsilon n_1 - f n_2, f n_1 + \epsilon n_2)/(\epsilon^2 + f^2)^{1/2}$. Therefore, no-normal-flow (47) and no-heat-flux boundary conditions (41)–(43) are natural and proper boundary conditions for the PG model (34)–(37). Since (47) is in fact a boundary condition on the temperature, thus, any additional and incompatible boundary condition to the temperature, on top of (41)–(43) would make the fourth-order diffusion $\nabla \cdot q$ overdetermined. Finally, the model is supplemented with the initial condition:

$$T(x, y, z, 0) = T_0(x, y, z), \tag{48}$$

where $T_0$ is a given function.

We observe that in the case when $\beta = 0$, i.e. $f = f_0$, our 'horizontal' hyper-diffusion term $\nabla \cdot q(T)$ reduces to the form $\lambda \Delta^2 T - \mu \Delta \partial_z^2 T - \kappa_h \Delta T$, which is in the spirit of the hyper-diffusion term in (30) that was proposed by Samelson and Vallis in [23, 24]. However, due to the fact that $\beta \neq 0$ our proposed hyper-diffusion term takes a more involved form, which is necessary to guarantee the dissipativity of this operator under the given physical boundary conditions (38)–(43). It is worth adding that the present PG formulation (34)–(43) has been explored with some preliminary computations by Samelson [20] using a modified version of the $\beta$-plane numerical code developed in [24]. In the $\beta$-plane case (for which the horizontal coordinates are Cartesian $x$ and $y$, and the Coriolis parameter $f = f_0 + \beta y$, with $f_0$ and $\beta$ constant), the additional horizontal diffusion terms in the thermodynamic equation reduce to a single term proportional to $T_{xx}$. The preliminary computations indicate that inclusion of these additional terms only slightly modifies the previous numerical solutions reported in [24] which uses hyper-diffusion term of the form suggested in (30). With the rigorous analytical results proved here for the modified system (34)–(43), this provides new theoretical and mathematical support for the approach and analysis of [23]. Note that these preliminary computations did not include the additional mixed horizontal–vertical diffusion term $\mu \Delta T_{zz}$, which is a crucial term for our rigorous mathematical analysis.

## 2. Preliminaries

It is natural to assume that $T^*$ satisfies the compatibility boundary conditions:

$$\frac{\partial T^*}{\partial \vec{e}} = 0, \qquad \text{on } \partial M. \tag{49}$$

$$q(T^*) \cdot \vec{n} = 0, \qquad \text{on } \partial M. \tag{50}$$

Let $\tilde{T} = T - T^*$. Due to the compatibility boundary conditions (49) and (50), it is clear that $\tilde{T}$ satisfies the following homogeneous boundary conditions:

$$\left.\frac{\partial \tilde{T}}{\partial z}\right|_{z=-h} = 0, \qquad \left.\left(\frac{\partial \tilde{T}}{\partial z} + \frac{\alpha}{K_v}\tilde{T}\right)\right|_{z=0} = 0, \qquad \left.\frac{\partial \tilde{T}}{\partial \vec{e}}\right|_{\Gamma_s} = 0, \qquad q(\tilde{T}) \cdot \vec{n}|_{\Gamma_s} = 0. \tag{51}$$

By replacing $T$ by $\tilde{T}$, we have the following new formulation of the system (34)–(48):

$$\partial_t \tilde{T} + \nabla \cdot q(\tilde{T}) - K_v \tilde{T}_{zz} + v \cdot \nabla \tilde{T} + w \partial_z \tilde{T} + v \cdot \nabla T^* = Q^*, \tag{52}$$

$$\left.\left(\partial_z \tilde{T} + \frac{\alpha}{K_v}\tilde{T}\right)\right|_{z=0} = 0, \qquad \partial_z \tilde{T}|_{z=-h} = 0, \qquad \left.\frac{\partial \tilde{T}}{\partial \vec{e}}\right|_{\Gamma_s} = 0, \qquad q(\tilde{T}) \cdot \vec{n}|_{\Gamma_s} = 0, \tag{53}$$

$$\tilde{T}(x, y, z, 0) = T_0(x, y, z) - T^*(x, y). \tag{54}$$

From now on $q(\tilde{T})$ is given by (32). Here $v$ and $w$ are determined by the use of (22) and (36), and the fact that the average of $v$ in the $z$-direction is zero. This can be easily seen by integration of (34) and (35), which yields the following system:

$$\nabla \bar{p} + f\vec{k} \times \bar{v} + \epsilon \bar{v} = 0, \tag{55}$$

$$\nabla \cdot \bar{v} = 0, \tag{56}$$

$$\bar{v} \cdot \vec{n} = 0. \tag{57}$$

Here $\bar{v}$ and $\bar{p}$ are the averages of $v$ and $p$ in the $z$-direction. Hence, multiplying by $\bar{v}$ and integrating over $M$, we obtain that $\bar{v} = 0$. The exact expressions of $v_1$, $v_2$ and $w$ in terms of $T$ are:

$$v_1 = \int_{-h}^{z} \frac{\epsilon \tilde{T}_x(x, y, \xi, t) + f \tilde{T}_y(x, y, \xi, t)}{\epsilon^2 + f^2} \, \mathrm{d}\xi + \left(z + \frac{h}{2}\right) \frac{\epsilon T_x^*(x, y) + f T_y^*(x, y)}{\epsilon^2 + f^2} \\ - \frac{1}{h} \int_{-h}^{0} \int_{-h}^{\eta} \frac{\epsilon \tilde{T}_x(x, y, \xi, t) + f \tilde{T}_y(x, y, \xi, t)}{\epsilon^2 + f^2} \, \mathrm{d}\xi \, \mathrm{d}\eta, \tag{58}$$

$$v_2 = \int_{-h}^{z} \frac{-f \tilde{T}_x(x, y, \xi, t) + \epsilon \tilde{T}_y(x, y, \xi, t)}{\epsilon^2 + f^2} \, \mathrm{d}\xi + \left(z + \frac{h}{2}\right) \frac{-f T_x^*(x, y) + \epsilon T_y^*(x, y)}{\epsilon^2 + f^2} \\ - \frac{1}{h} \int_{-h}^{0} \int_{-h}^{\eta} \frac{-f \tilde{T}_x(x, y, \xi, t) + \epsilon \tilde{T}_y(x, y, \xi, t)}{\epsilon^2 + f^2} \, \mathrm{d}\xi \, \mathrm{d}\eta. \tag{59}$$

Using (36), (38) and (39) we have

$$w(x, y, z) = -\int_{-h}^{z} (\nabla \cdot v)(x, y, \xi) \, \mathrm{d}\xi. \tag{60}$$

From (58)–(60) we obtain

$$
\begin{aligned}
w = &-\int_{-h}^{z}\int_{-h}^{\eta}\Bigg[\frac{\epsilon\Delta\tilde{T}(x,y,\xi,t) - f_0\tilde{T}_x(x,y,\xi,t)}{\epsilon^2+f^2} \\
&+\frac{2f_0 f}{(\epsilon^2+f^2)^2}(-f\tilde{T}_x(x,y,\xi,t)+\epsilon\tilde{T}_y(x,y,\xi,t))\Bigg]\mathrm{d}\xi\,\mathrm{d}\eta \\
&+\frac{z+h}{h}\int_{-h}^{0}\int_{-h}^{\eta}\Bigg[\frac{\epsilon\Delta\tilde{T}(x,y,\xi,t) - f_0\tilde{T}_x(x,y,\xi,t)}{\epsilon^2+f^2} \\
&-\frac{2f_0 f}{(\epsilon^2+f^2)^2}(-f\tilde{T}_x(x,y,\xi,t)+\epsilon\tilde{T}_y(x,y,\xi,t))\Bigg]\mathrm{d}\xi\,\mathrm{d}\eta \\
&-\frac{z(z+h)}{2}\Bigg[\frac{\epsilon\Delta T^*(x,y) - f_0 T_x^*(x,y)}{\epsilon^2+f^2} \\
&+\frac{2f_0 f}{(\epsilon^2+f^2)^2}(-fT_x^*(x,y)+\epsilon T_y^*(x,y))\Bigg]. \qquad (61)
\end{aligned}
$$

## 2.1. *Functional spaces and weak formulation*

We denote by $L^p(\Omega)$ and $H^m(\Omega)$ the standard $L^p$-spaces and Sobolev spaces, respectively. Following the notations in [2, 21], we set

$$
\mathcal{V} = \left\{R \in C^\infty(\bar{\Omega}) : \left(\partial_z\tilde{T} + \frac{\alpha}{K_v}\tilde{T}\right)\Bigg|_{z=0} = 0;\ \ \partial_z\tilde{T}|_{z=-h} = 0;\ \ q(\tilde{T})\cdot\vec{n}|_{\Gamma_s} = 0\right\}. \qquad (62)
$$

For every $R \in C^\infty(\bar{\Omega})$, denote by

$$
\|R\|_{V_2} = (\|R\|^2_{H^1(\Omega)} + \|\nabla R\|^2_{H^1(\Omega)})^{1/2} \qquad (63)
$$

and

$$
\|R\|_{V_4} = (\|R\|^2_{H^2(\Omega)} + \|\Delta R\|^2_{H^2(\Omega)})^{1/2}. \qquad (64)
$$

Also, we denote by

$$
V_2 = \text{the closure of } \mathcal{V} \text{ with respect to the topology induced by the norm } \|\cdot\|_{V_2} \qquad (65)
$$

and

$$
V_4 = \text{the closure of } \mathcal{V} \text{ with respect to the topology induced by the norm } \|\cdot\|_{V_4}. \qquad (66)
$$

It is clear that $V_2$ and $V_4$ are separable Hilbert spaces. Next we define the bilinear form $a : V_2 \times V_2 \to \mathbb{R}$ is defined by

$$
\begin{aligned}
a(R_1,R_2) = \alpha\int_{z=0}\left[R_1\,R_2 + \frac{\mu}{K_v}\nabla R_1\cdot\nabla R_2\right]\mathrm{d}x\,\mathrm{d}y + \int_\Omega [K_h\nabla R_1\cdot\nabla R_2 + K_v(R_1)_z\,(R_2)_z \\
+\lambda\nabla\cdot(H^T\nabla R_1)\,\nabla\cdot(H^T\nabla R_2) + \mu\nabla(R_1)_z\cdot\nabla(R_2)_z]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \qquad (67)
\end{aligned}
$$

We will denote by

$$
|R|^2 = \int_\Omega |R|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \qquad (68)
$$

$$
\begin{aligned}
\|R\|^2 = a(R,R) = \alpha\int_{z=0}\left[|R|^2 + \frac{\mu}{K_v}|\nabla R|^2\right]\mathrm{d}x\,\mathrm{d}y \\
+\int_\Omega [K_h|\nabla R|^2 + K_v|\partial_z R|^2 + \lambda|\nabla\cdot(H^T\nabla R)|^2 + \mu|\nabla R_z|^2]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \qquad (69)
\end{aligned}
$$

Now we give the definition of weak and strong solutions to the model.

**Definition 1.** *Let S be any fixed positive time. A function* $\tilde{T}(x, y, z, t)$ *is called a weak solution of* (52)–(54) *on* $[0, S]$ *if*

$$\tilde{T} \in C_w([0, S], L^2(\Omega)) \cap L^2([0, S], V_2), \qquad \tilde{T}_t \in L^1([0, S], V_2')$$

*and*

$$\begin{aligned}\int_\Omega \tilde{T}(t)\psi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \int_\Omega \tilde{T}(t_0)\psi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{t_0}^t a(\tilde{T}(s), \psi)\,\mathrm{d}s \\ + \int_{t_0}^t \int_\Omega [v \cdot \nabla \tilde{T}(s) + w\tilde{T}_z(s) + v \cdot \nabla T^*]\psi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s \\ = \int_{t_0}^t \int_\Omega Q^*\psi \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}s \end{aligned} \tag{70}$$

*for every* $\psi \in V_2$. *Here* $C_w([0, S], L^2(\Omega))$ *is the functional space of all weakly continuous functions from* $[0, S]$ *to* $L^2(\Omega)$. *Furthermore,* $\tilde{T}(x, y, z, t)$ *is a strong solution of (52)–(54) on* $[0, S]$ *if it is a weak solution and*

$$\tilde{T}(x, y, z, t) \in C([0, S], V_2) \cap L^2([0, S], V_4).$$

Next, we give some remarks about the following boundary value problem:

$$\begin{aligned}&\nabla \cdot q(R) - K_v R_{zz} = g(x, y, z) \qquad \text{in} \quad \Omega, \\ &\left(\partial_z R + \frac{\alpha}{K_v} R\right)\Bigg|_{z=0} = 0, \qquad \partial_z R|_{z=-h} = 0, \qquad \frac{\partial R}{\partial \vec{e}}\Bigg|_{\Gamma_s} = 0, \qquad q(R) \cdot \vec{n}|_{\Gamma_s} = 0,\end{aligned} \tag{71}$$

where $q(R)$ is given by (32). By integrating by parts and the boundary conditions (51), we have

$$\begin{aligned}\int_\Omega R_1(\nabla \cdot q(R_2) - K_v(R_2)_{zz})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ = \int_\Omega R_2(\nabla \cdot q(R_1) + \partial_z q_3(R_1))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = a(R_1, R_2)\end{aligned} \tag{72}$$

for all $R_1, R_2 \in \tilde{\mathcal{V}}$. Note that

$$\vec{e} \cdot \vec{n} = \epsilon \neq 0.$$

Namely, the vector $\vec{e}$ is not tangent to $\partial M$. Using the classical regularity results in smooth domains, $M \subset \mathbb{R}^2$, for the Laplacian operator with oblique boundary condition, we have the following proposition.

**Proposition 2.** *Suppose that* $R(\cdot, \cdot, z)$ *satisfies the boundary condition* $\partial R/\partial \vec{e}|_{\Gamma_s} = 0$ *and that* $\Delta R(\cdot, \cdot, z)$ *is in* $L^2(M)$ *for every fixed z. Then there exists a constant* $C > 0$ *(independent of z) such that*

$$\|R(\cdot, \cdot, z)\|_{H^2(M)}^2 \leqslant C[\|\Delta R(\cdot, \cdot, z)\|_{L^2(M)}^2 + \|R(\cdot, \cdot, z)\|_{L^2(M)}^2]. \tag{73}$$

*Consequently, there is a constant* $C_0 > 0$ *such that*

$$\frac{1}{C_0}\|R\|^2 \leqslant \|R\|_{V_2}^2 \leqslant C_0\|R\|^2. \tag{74}$$

**Proof.** As we mentioned earlier the proof of (73) is a result of classical regularity theory of elliptic equations. Note that since $\partial R/\partial \vec{e}|_{\Gamma_s} = 0$ (thanks to (71)) we also have $\partial R_z/\partial \vec{e}|_{\Gamma_s} = 0$. Now applying (73) to $\partial_z R$ instead of $R$ with the corresponding boundary condition $\partial R_z/\partial \vec{e}|_{\Gamma_s} = 0$ implies (74). □

Using (74), Lax–Milgram theorem shows that there is a unique solution $R \in V_2$ for the boundary value problem (71) satisfying

$$\|R\| \leqslant |g|. \tag{75}$$

Moreover, using integration by parts in (71) leads to

$$|g|^2 = \int_\Omega [|\nabla \cdot q(R)|^2 + K_v^2|R_{zz}|^2 + 2K_v K_h|\nabla R_z|^2 + 2K_v\mu|\nabla R_{zz}|^2 \\ +2K_v\lambda|\nabla \cdot (H^T\nabla R_z)|^2]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\ +2\alpha\int_{z=0}[K_h|\nabla R|^2 + 2\lambda|\nabla \cdot (H^T\nabla R)|^2]\,\mathrm{d}x\,\mathrm{d}y.$$

As a result, we have

$$|R_{zz}| \leqslant C|g|. \tag{76}$$

Note that the principle part of operator $\nabla\cdot q(R) - K_v R_{zz}$ is $-\lambda\Delta(-\Delta R + R) - K_v(-\Delta R + R)_{zz}$. Then, using a symmetry argument in the $z$-direction and the standard regularity results for the Laplacian operator (see, e.g. [9], p 89), we get

$$\| -\Delta R + R\|_{H^2(\Omega)} \leqslant C|g|.$$

Therefore,

$$\|R\|_{V_4} = (\|\Delta R\|^2_{H^2(\Omega)} + \|R\|^2_{H^2(\Omega)})^{1/2} \leqslant C|g|. \tag{77}$$

Therefore, using proposition 2, (74) and Rellich lemma [1], one can show that the operator $\nabla\cdot q(R) - K_v R_{zz}$ with domain $V_4$ is a positive self-adjoint operator with compact inverse. Thus, the space $L^2(\Omega)$ possesses an orthonormal basis $\{\phi_k(x, y, z)\}_{k=1}^\infty$ of eigenfunctions of the operator $\nabla\cdot q(R) - K_v R_{zz}$, such that

$$\nabla\cdot q(\phi_k) - K_v(\phi_k)_{zz} = \lambda_k\phi_k, \tag{78}$$

where $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, and $\lim_{k\to\infty}\lambda_k = \infty$. Moreover, using standard results(cf, e.g., [3]), we have

$$\frac{k}{C_1} \leqslant \frac{\lambda_k}{\lambda_1}. \tag{79}$$

We will denote by $H_m = \mathrm{span}\{\phi_1, \ldots, \phi_m\}$, and by $P_m : L^2(\Omega) \to H_m$ the $L^2(\Omega)$ orthogonal projection onto $H_m$. Hereafter, $C$, which may depend on the domain $\Omega$ and the constant parameters $\epsilon, f_0, \beta, \alpha, K_h, K_v, \lambda, \mu$ in the system (52)–(54), will denote a constant that may change from line to line.

## 3. Global existence, uniqueness and well-posedness of weak solutions

Now we are ready to show the global existence and uniqueness of weak solutions to the system (52)–(54).

**Theorem 3.** *Suppose that* $T^* \in H^2(M)$ *and* $Q \in L^2(\Omega)$. *Then for every* $T_0(x, y, z) \in L^2(\Omega)$, *and* $S > 0$, *there is a unique weak solution* $\tilde{T}$ *of the system* (52)–(54). *Moreover,* $\tilde{T}$ *satisfies*

$$\partial_t\tilde{T} \in L^{4r/(3r-1)}(0, S; V_2'), \qquad \forall r > 1, \\ |\tilde{T}(\sigma)|^2 + \int_0^\sigma \|\tilde{T}(s)\|^2\,\mathrm{d}s \leqslant K_2(S, Q, \tilde{T}_0, T^*), \qquad \forall\sigma \in [0, S], \tag{80}$$

*where* $V_2'$ *is the dual space of* $V_2$, $K_1(S, Q, \tilde{T}_0, T^*)$ *and* $K_2(S, Q, \tilde{T}_0, T^*)$ *are as specified in (90) and (92), respectively.*

**Proof.** First, let us prove the existence of the weak solution for system (52)–(54). We will use a Galerkin-like procedure, based on the eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$, to show the existence. Let $m \in \mathbb{Z}^+$ be fixed, the Galerkin approximating system of order $m$ that we use for (52)–(54) reads:

$$\frac{\partial}{\partial t}\tilde{T}_m + \nabla \cdot q(\tilde{T}_m) - K_v(\tilde{T}_m)_{zz} + P_m\left[v \cdot \nabla \tilde{T}_m + w\frac{\partial \tilde{T}_m}{\partial z} + (v \cdot \nabla)T^*\right] = P_m Q^*, \tag{81}$$

$$\tilde{T}_m(x, y, z, 0) = P_m[T_0(x, y, z) - T^*(x, y)], \tag{82}$$

where $\tilde{T}_m = \sum_{k=1}^m a_k(t)\phi_k(x, y, z)$, and $v(\tilde{T}_m) = (v_1, v_2)$, $w = w(\tilde{T}_m)$ are given explicitly with the help of (58), (59) and (61), and

$$q(\tilde{T}_m) = -K_h\nabla\tilde{T}_m + \lambda H\nabla(\nabla \cdot (H^T\nabla\tilde{T}_m)) + \mu\nabla(\tilde{T}_m)_{zz}. \tag{83}$$

We stress again that $v$ and $w$ depend on $m$ since they are functions of $\tilde{T}_m$. However, we will drop the explicit dependence $m$ to simplify the notation. Equation (81) is an ODE system with the unknown $a_k(t), k = 1, \ldots, m$. Furthermore, it is easy to check that each term of equation (81) is locally Lipschitz in $\tilde{T}_m$. Therefore, there is a unique solution $a_k(t), k = 1, \ldots, m$, to the equation (81) for a short interval of time $[0, S^*)$. By taking the $L^2(\Omega)$ inner product of equation (81) with $\tilde{T}_m$, we easily obtain

$$\frac{1}{2}\frac{\mathrm{d}|\tilde{T}_m|^2}{\mathrm{d}t} + \|\tilde{T}_m\|^2 + \int_\Omega (v \cdot \nabla T^*)\,\tilde{T}_m\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_\Omega \tilde{T}_m Q^*\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{84}$$

Furthermore, using Sobolev-type inequalities, we have

$$\left|\int_\Omega (v \cdot \nabla T^*)\tilde{T}_m\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| \leqslant \frac{1}{4}\|\tilde{T}_m\|^2 + C\|T^*\|_{H^2(M)}^2(1 + |\tilde{T}_m|^2). \tag{85}$$

Applying Cauchy–Schwarz inequality and the definition of $Q^*$, we obtain

$$\left|\int_\Omega Q^*\tilde{T}_m\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| \leqslant \frac{1}{2}|Q|^2 + \frac{1}{2}|\tilde{T}_m|^2 + C\|T^*\|_{H^2(M)}^2 + \frac{1}{4}\|\tilde{T}_m\|^2. \tag{86}$$

Therefore,

$$\frac{\mathrm{d}|\tilde{T}_m|^2}{\mathrm{d}t} + \|\tilde{T}_m\|^2 \leqslant C\big[|Q|^2 + \|T^*\|_{H^2(M)}^2\big] + C\|T^*\|_{H^2(M)}^2|\tilde{T}_m|^2. \tag{87}$$

Thanks to Gronwall inequality, we conclude

$$|\tilde{T}_m(t)|^2 \leqslant [|T_0|^2 + C(|Q|^2 + \|T^*\|_{H^2(M)}^2)]\exp(C\|T^*\|_{H^2(M)}^2 t) \tag{88}$$

when $0 \leqslant t < S^*$. But since the right-hand side is bounded as $t$ goes to $S^*$, we conclude that $\tilde{T}_m(t)$ must exist globally, i.e. $S^* = +\infty$. Therefore, for any given $S > 0$ and any $t \in [0, S]$, we have

$$|\tilde{T}_m(t)|^2 \leqslant K_1(S, Q, \tilde{T}_0, T^*), \tag{89}$$

where

$$K_1(S, Q, \tilde{T}_0, T^*) = [|T_0|^2 + C(|Q|^2 + \|T^*\|_{H^2(M)}^2)]\exp(C\|T^*\|_{H^2(M)}^2 S). \tag{90}$$

By integrating (87) with respect to $t$ over $[0, S]$, and by (89), we get

$$\int_0^S \|\tilde{T}_m\|^2\,\mathrm{d}s \leqslant K_2(S, Q, \tilde{T}_0, T^*), \tag{91}$$

where

$$K_2(S, Q, \tilde{T}_0, T^*) = |T_0|^2 + C[|Q|^2 + \|T^*\|_{H^2(M)}^2 + \|T^*\|_{H^2(M)}^2 K_1(S, Q, |T_0|, T^*)]S \tag{92}$$

and $K_1(S, Q, \tilde{T}_0, T^*)$ is as in (90). Note that the estimate (89) is unbounded in time (i.e. as $S \to \infty$), but it is uniformly bounded in $m$. However, in section 5, we will present a sharper estimate which is asymptotically bounded in time. As a result of all the above we have $\tilde{T}_m$ exists globally in time and is uniformly bounded, in $m$, in $L^\infty([0, S]; L^2(\Omega))$ and $L^2([0, S]; V_2)$ norms.

Next, let us show that $\partial_t \tilde{T}_m$ is uniformly bounded, in $m$, in the $L^{4r/(3r-1)}([0, S]; V_2')$ norm for every $r > 1$. From (81), we have, for every $\psi \in V_2$

$$\left\langle \frac{\partial}{\partial t}\tilde{T}_m, \psi \right\rangle = \left\langle P_m Q^* - K_v(\tilde{T}_m)_{zz} + \nabla \cdot q(\tilde{T}_m) - P_m \left[ v \cdot \nabla \tilde{T}_m + w \frac{\partial \tilde{T}_m}{\partial z} + (v \cdot \nabla) T^* \right], \psi \right\rangle.$$

Here, $\langle \cdot, \cdot \rangle$ is the dual action of $V_2'$. It is clear that

$$|\langle P_m Q^*, \psi \rangle| \leqslant |\langle P_m Q, \psi \rangle| + |\langle \nabla \cdot q(T^*), \psi \rangle| \leqslant \|Q\||\psi| + C\|T^*\|_{H^2(M)}\|\psi\| \tag{93}$$

and by integration by parts we have

$$|\langle -K_v(\tilde{T}_m)_{zz} + \nabla \cdot q(\tilde{T}_m), \psi \rangle| \leqslant C\|\tilde{T}_m\|\|\psi\|. \tag{94}$$

Next, we note that

$$\left| \left\langle P_m \left[ v \cdot \nabla(\tilde{T}_m + T^*) + w \frac{\partial \tilde{T}_m}{\partial z} \right], \psi \right\rangle \right| = \left| \int_\Omega \left[ v \cdot \nabla(\tilde{T}_m + T^*) + w \frac{\partial \tilde{T}_m}{\partial z} \right] \psi_m \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \right|,$$

where $\psi_m = P_m \psi$. Thus, by integration by parts, we obtain

$$\left| \left\langle P_m \left[ v \cdot \nabla(\tilde{T}_m + T^*) + w \frac{\partial \tilde{T}_m}{\partial z} \right], \psi \right\rangle \right| = \left| \int_\Omega \left[ v \cdot \nabla \psi_m + w \frac{\partial \psi_m}{\partial z} \right] (\tilde{T}_m + T^*) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \right| \tag{95}$$

and since

$$\left| \int_\Omega (v \cdot \nabla \psi_m)(\tilde{T}_m + T^*) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \right| \leqslant \|\nabla \psi_m\|_{L^6(\Omega)} \|v\|_{L^3(\Omega)} |\tilde{T}_m + T^*|,$$

we have, using Sobolev inequalities

$$\left| \int_\Omega (v \cdot \nabla \psi_m)(\tilde{T}_m + T^*) \right| \leqslant C[\|\tilde{T}_m\| + \|T^*\|_{H^2(M)}][|\tilde{T}_m| + \|T^*\|_{L^2(M)}]\|\psi_m\|. \tag{96}$$

For the other term in (95), we use (60) to get

$$\begin{aligned}
\left| \int_\Omega w \partial_z \psi_m (\tilde{T}_m + T^*) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \right| &= \left| \int_\Omega \left( \int_{-h}^z \nabla \cdot v \, \mathrm{d}\xi \right) \partial_z \psi_m (\tilde{T}_m + T^*) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \right| \\
&\leqslant \int_\Omega \left( \int_{-h}^0 |\nabla \cdot v(x, y, \xi, t)| \, \mathrm{d}\xi \right) |\partial_z \psi_m(x, y, z)||\tilde{T}_m(x, y, z, t) \\
&\quad + T^*(x, y)| \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\
&= \int_M \left[ \int_{-h}^0 |\nabla \cdot v(x, y, \xi, t)| \, \mathrm{d}\xi \int_{-h}^0 |\partial_z \psi_m(x, y, z)||\tilde{T}_m(x, y, z, t) \right. \\
&\quad \left. + T^*(x, y)| \, \mathrm{d}z \right] \mathrm{d}x \, \mathrm{d}y.
\end{aligned}$$

Using Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
&\int_{-h}^0 |\partial_z \psi_m(x, y, z)||\tilde{T}_m(x, y, z, t) + T^*(x, y)| \, \mathrm{d}z \\
&\qquad \leqslant \left( \int_{-h}^0 |\partial_z \psi_m(x, y, z)|^2 \, \mathrm{d}z \right)^{1/2} \left( \int_{-h}^0 |\tilde{T}_m(x, y, z, t) + T^*(x, y)|^2 \, \mathrm{d}z \right)^{1/2}.
\end{aligned}$$

Thus,

$$\left|\int_\Omega w\partial_z\psi_m(\tilde T_m+T^*)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|$$

$$\leqslant \int_M\left(\int_{-h}^0|\nabla\cdot v(x,y,\xi,t)|\,\mathrm{d}\xi\right)\left(\int_{-h}^0|\partial_z\psi_m(x,y,z)|^2\,\mathrm{d}z\right)^{1/2}$$
$$\times\left(\int_{-h}^0|\tilde T_m(x,y,z,t)+T^*(x,y)|^2\,\mathrm{d}z\right)^{1/2}\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant\left(\int_M\left(\int_{-h}^0|\nabla\cdot v(x,y,\xi,t)|\,\mathrm{d}\xi\right)^2\mathrm{d}x\,\mathrm{d}y\right)^{1/2}$$
$$\times\left(\int_M\left(\int_{-h}^0|\partial_z\psi_m(x,y,z)|^2\,\mathrm{d}z\right)^{r'}\mathrm{d}x\,\mathrm{d}y\right)^{1/2r'}$$
$$\times\left(\int_M\left(\int_{-h}^0|\tilde T_m(x,y,z,t)+T^*(x,y)|^2\,\mathrm{d}z\right)^{r}\mathrm{d}x\,\mathrm{d}y\right]^{1/2r}. \tag{97}$$

By using Minkowsky inequality

$$\left[\int_M\left(\int_{-h}^0|\tilde T_m(x,y,z,t)+T^*(x,y)|^2\,\mathrm{d}z\right)^{r}\mathrm{d}x\,\mathrm{d}y\right]$$
$$\leqslant\left(\int_{-h}^0\left(\int_M|\tilde T_m(x,y,z,t)+T^*(x,y)|^{2r}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/r}\mathrm{d}z\right)^{r}.$$

Using Sobolev inequalities, we have for every fixed $z$ and $t$

$$\left(\int_M|\tilde T_m(x,y,z,t)+T^*(x,y)|^{2r}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2r}$$
$$\leqslant C_r\|\tilde T_m(z,t)+T^*\|_{L^2(M)}^{(1/2)+(1/2r)}\|\tilde T_m(z,t)+T^*\|_{H^2(M)}^{(1/2)-(1/2r)}$$

and therefore

$$\left(\int_{-h}^0\left(\int_M|\tilde T_m(x,y,z,t)+T^*|^{2r}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/r}\mathrm{d}z\right)^{1/2}$$
$$\leqslant C_r\left(\int_{-h}^0\|\tilde T_m(z,t)+T^*\|_{L^2(M)}^2\,\mathrm{d}z\right)^{(r+1)/4r}$$
$$\times\left(\int_{-h}^0\|\tilde T_m(z,t)+T^*\|_{H^2(M)}^2\,\mathrm{d}z\right)^{(r-1)/4r}$$
$$\leqslant C_r|\tilde T_m+T^*|^{(r+1)/2r}\|\tilde T_m+T^*\|_{V_2}^{(r-1)/2r}. \tag{98}$$

Similarly,

$$\left(\int_M\left(\int_{-h}^0|\partial_z\psi_m(x,y,z)|^2\,\mathrm{d}z\right)^{r'}\mathrm{d}x\,\mathrm{d}y\right)^{1/2r'}$$
$$\leqslant\left(\int_{-h}^0\left(\int_M|\partial_z\psi_m(x,y,z)|^{2r'}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/r'}\mathrm{d}z\right)^{1/2}$$
$$\leqslant C_r\left(\int_{-h}^0\|\partial_z\psi_m\|_{H^1(M)}^2\,\mathrm{d}z\right)^{1/2}\leqslant C_r\|\psi_m\|$$

and

$$\left|\int_\Omega w\partial_z\psi_m(\tilde{T}_m+T^*)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|$$
$$\leqslant C_r(|\tilde{T}_m|+\|T^*\|_{L^2(M)})^{(r+1)/2r}(\|\tilde{T}_m\|+\|T^*\|_{H^2(M)})^{(3r-1)/2r}\|\psi_m\|. \tag{99}$$

By (96) and (99), we have

$$\left|\left\langle P_m\left[v\cdot\nabla(\tilde{T}_m+T^*)+w\frac{\partial\tilde{T}_m}{\partial z}\right],\psi\right\rangle\right|$$
$$\leqslant C_r(|\tilde{T}_m|+\|T^*\|_{L^2(M)})^{(r+1)/2r}(\|\tilde{T}_m\|+\|T^*\|_{H^2(M)})^{(3r-1)/2r}\|\psi_m\|.$$

Since $\psi\in V_2$, then the Fourier series

$$\sum_{k=1}^{\infty}\left(\int_\Omega\psi\phi_k\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)\phi_k=\psi_m+\sum_{k=m+1}^{\infty}\left(\int_\Omega\psi\phi_k\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right)\phi_k$$

converges to $\psi$ in $V_2$ (cf, [9], p 64). Thus, $\|\psi_m\|_{V_2}\leqslant C\|\psi\|_{V_2}$, and by proposition 2, we have

$$\left|\left\langle P_m\left[v\cdot\nabla(\tilde{T}_m+T^*)+w\frac{\partial\tilde{T}_m}{\partial z}\right],\psi\right\rangle\right|$$
$$\leqslant C_r(|\tilde{T}_m|+\|T^*\|_{L^2(M)})^{(r+1)/2r}(\|\tilde{T}_m\|+\|T^*\|_{H^2(M)})^{(3r-1)/2r}\|\psi\| \tag{100}$$

for every $r>1$. Moreover, the estimates (93)–(100), (89) and (91) yield

$$|\langle\partial_t\tilde{T}_m,\psi\rangle|\leqslant C(|Q|+\|T^*\|_{H^2(M)})\|\psi\|+C\|\tilde{T}_m\|\|\psi\|$$
$$+C_r(|\tilde{T}_m|+\|T^*\|_{L^2(M)})^{(r+1)/2r}(\|\tilde{T}_m\|+\|T^*\|_{H^2(\Omega)})^{(3r-1)/2r}\|\psi\|.$$

Thus, due to (89) and (91), we have

$$\int_0^S\|\partial_t\tilde{T}_m(t)\|_{V_2'}^{4r/(3r-1)}\,\mathrm{d}t\leqslant K_3(S,Q,\tilde{T}_0,T^*,r), \tag{101}$$

where

$$K_3(S,Q,\tilde{T}_0,T^*,r)=C[1+|Q|+\|T^*\|_{H^2(M)}]^{4r/(3r-1)}S+CK_2(S,Q,|T_0|,T^*)$$
$$+C_r[K_1(S,Q,\tilde{T}_0,T^*)+\|T^*\|_{H^2(\Omega)}]^{(2(r+1))/(3r-1)}$$
$$\times[K_2(S,Q,\tilde{T}_0,T^*)+\|T^*\|_{H^2(M)}^2S]. \tag{102}$$

Therefore, $\partial_t\tilde{T}_m$ is uniformly bounded, in $m$, in the $L^{4r/(3r-1)}([0,S];V_2')$ norm, for every $r>1$. Thanks to (89), (91) and (101), one can apply the Aubin's compactness theorem (cf, e.g. [6,11,26]) and extract a subsequence $\{\tilde{T}_{m_j}\}$ of $\{\tilde{T}_m\}$; a subsequence $\{v_{m_j}\}$ of $\{v_m=v(\tilde{T}_m)\}$ and a subsequence $\{\partial_t\tilde{T}_{m_j}\}$ of $\{\partial_t\tilde{T}_m\}$; which converge to $\tilde{T}\in L^\infty([0,S];L^2(\Omega))\cap L^2([0,S];V_2)$ and $\partial_t\tilde{T}\in L^{4r/(3r-1)}([0,S];V_2')$, respectively, in the following sense:

$$\tilde{T}_{m_j}\to\tilde{T}\qquad\text{in }\ L^\infty([0,S];L^2(\Omega)),$$
$$\tilde{T}_{m_j}\to\tilde{T}\qquad\text{in }\ L^2([0,S];H^1(\Omega))\text{ weakly},$$
$$\partial_t\tilde{T}_{m_j}\to\partial_t\tilde{T}\qquad\text{in }\ L^{4r/(3r-1)}([0,S];V_2')\text{ weakly}.$$

Note that since $\tilde{T}_{m_j}\in\mathcal{V}$, by integration by parts it is clear that

$$\int_\Omega\tilde{T}_{m_j}(x,y,z,t)P_{m_j}\psi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z-\int_\Omega\tilde{T}_{m_j}(x,y,z,t_0)P_{m_j}\psi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z+a(\tilde{T}_{m_j},P_{m_j}\psi)$$
$$+\int_{t_0}^t\int_\Omega[(v_{m_j}\cdot\nabla\tilde{T}_{m_j})P_{m_j}\psi+w_{m_j}\partial_z\tilde{T}_{m_j}P_{m_j}\psi+(v_{m_j}\cdot\nabla T^*)P_{m_j}\psi]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$=\int_\Omega Q^*P_{m_j}\psi\ \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

for every $\psi \in C^\infty([0, S]; V_2)$, and for every $t, t_0 \in [0, S]$. By passing to the limit, one can show as in the case of Navier–Stokes equations (see, e.g. [6, 26]) that $\tilde{T}$ also satisfies (70). In other words, $\tilde{T}$ is a weak solution of the system (52)–(54).

Next, we show the uniqueness of the weak solution. Let $\tilde{T}_1$ and $\tilde{T}_2$ be two weak solutions of the system (52)–(54) with respect to initial values $\tilde{T}_0'$ and $\tilde{T}_0''$, and let $(v' = (v_1', v_2'), w')$ and $(v'' = (v_1'', v_2''), w'')$ be given by (58)–(61) with respect to $\tilde{T}_1$ and $\tilde{T}_2$. Denote by $u = v' - v''$, $u_3 = w' - w''$ and $\chi = \tilde{T}_1 - \tilde{T}_2$. Unfortunately, since we only have

$$\partial_t \chi \in L^{4r/(3r-1)}(0, S; V_2'), \qquad \forall r > 1,$$

we are not able to apply the standard energy method and Lions lemma (cf [26] lemma 1.2, p 260). However, let us instead consider

$$\chi_m = P_m \chi \qquad \text{and} \qquad \tilde{\chi}_m = \chi - P_m \chi,$$

where $P_m$ is the $L^2(\Omega)$ orthogonal projection onto $H_m$. It is clear that $\chi_m$ satisfies the following equations:

$$\partial_t \chi_m + \nabla \cdot q(\chi_m) - K_v (\chi_m)_{zz} + P_m[u \cdot \nabla \tilde{T}_1 + u_3 \partial_z \tilde{T}_1 + v'' \cdot \nabla \chi + w'' \chi_z + u \cdot \nabla T^*] = 0, \tag{103}$$

$$\chi_m(x, y, z, 0) = P_m(\tilde{T}_0'(x, y, z) - \tilde{T}_0''(x, y, z)), \tag{104}$$

where $u = (u_1, u_2)$ and

$$q(\chi_m) = -K_h \nabla \chi_m + \lambda H \nabla(\nabla \cdot (H^T \nabla \chi_m)) + \mu \nabla(\chi_m)_{zz}, \tag{105}$$

$$\begin{aligned} u_1(x, y, z, t) = &\int_{-h}^{z} \frac{\epsilon \chi_x(x, y, \xi, t) + f \chi_y(x, y, \xi, t)}{\epsilon^2 + f^2}\, d\xi \\ &- \frac{1}{h} \int_{-h}^{0} \int_{-h}^{z} \frac{\epsilon \chi_x(x, y, \xi, t) + f \chi_y(x, y, \xi, t)}{\epsilon^2 + f^2}\, d\xi\, dz, \end{aligned} \tag{106}$$

$$\begin{aligned} u_2(x, y, z, t) = &\int_{-h}^{z} \frac{-f \chi_x(x, y, \xi, t) + \epsilon \chi_y(x, y, \xi, t)}{\epsilon^2 + f^2}\, d\xi \\ &- \frac{1}{h} \int_{-h}^{0} \int_{-h}^{z} \frac{-f \chi_x(x, y, \xi, t) + \epsilon \chi_y(x, y, \xi, t)}{\epsilon^2 + f^2}\, d\xi\, dz, \end{aligned} \tag{107}$$

$$\begin{aligned} u_3(x, y, z, t) = &- \int_{-h}^{z} \int_{-h}^{\eta} \left[ \frac{\epsilon \Delta \chi(x, y, \xi, t) - f_0 \chi_x(x, y, \xi, t)}{\epsilon^2 + f^2} \right. \\ &\left. + \frac{2 f_0 f}{(\epsilon^2 + f^2)^2} (-f \chi_x(x, y, \xi, t) + \epsilon \chi_y(x, y, \xi, t)) \right] d\xi\, d\eta \\ &+ \frac{z + h}{h} \int_{-h}^{0} \int_{-h}^{z} \left[ \frac{\epsilon \Delta \chi(x, y, \xi, t) - f_0 \chi_x(x, y, \xi, t)}{\epsilon^2 + f^2} \right. \\ &\left. - \frac{2 f_0 f}{(\epsilon^2 + f^2)^2} (-f \chi_x(x, y, \xi, t) + \epsilon \chi_y(x, y, \xi, t)) \right] d\xi\, dz. \end{aligned} \tag{108}$$

By taking the $V_2'$ dual action to equation (103) with $\chi_m$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d|\chi_m|^2}{dt} + \|\chi_m\|^2 &= -\langle u \cdot \nabla \tilde{T}_1 + u_3 \partial_z \tilde{T}_1 + v'' \cdot \nabla \chi + w'' \chi_z + u \cdot \nabla T^*, \chi_m \rangle \\ &= - \int_\Omega [u \cdot \nabla \tilde{T}_1 + u_3 \partial_z \tilde{T}_1 + v'' \cdot \nabla \chi + w'' \chi_z + u \cdot \nabla T^*] \chi_m\, dx\, dy\, dz. \end{aligned} \tag{109}$$

Next we estimate the equation (109) term by term

(i)

$$\left|\int_\Omega u\cdot\nabla T^*\chi_m\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| \leqslant C\|T^*\|_{H^2(M)}(|\chi_m|\|\chi_m\|+|\chi_m|\|\tilde{\chi}_m\|). \tag{110}$$

(ii) Using Hölder, Sobolev and Minkowsky inequalities, we obtain as in the existence proof

$$\left|\int_\Omega u\cdot\nabla\tilde{T}_1\chi_m\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| \leqslant C\|\tilde{T}_1\|(|\chi_m|\|\chi_m\|+|\tilde{\chi}_m|^{1/4}\|\tilde{\chi}_m\|^{3/4}|\chi_m|^{3/4}\|\chi_m\|^{1/4}). \tag{111}$$

(iii) Starting with (60), we have

$$u_3 = -\nabla\cdot\int_{-h}^{z} u(x,y,\xi,t)\,\mathrm{d}\xi.$$

Therefore,

$$\begin{aligned}\left|\int_\Omega u_3\,\partial_z\tilde{T}_1\,\chi_m\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| &\leqslant \left|\int_\Omega\left(\int_{-h}^0|u|\,\mathrm{d}z\right)|\nabla\partial_z\tilde{T}_1||\chi_m|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|\\ &\quad+\left|\int_\Omega|u||\nabla\chi_m||\tilde{T}_1|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|+\left|\int_\Omega\left(\int_{-h}^0|u|\,\mathrm{d}z\right)|\tilde{T}_1||\nabla\partial_z\chi_m|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|.\end{aligned}$$

Following the steps of getting estimate (111), we have

$$\begin{aligned}&\left|\int_\Omega\left(\int_{-h}^0|u|\,\mathrm{d}z\right)|\nabla\partial_z\tilde{T}_1||\chi_m|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|\\ &\quad=\left|\int_M\left(\int_{-h}^0|u|\,\mathrm{d}z\right)\left(\int_{-h}^0|\nabla\partial_z\tilde{T}_1||\chi_m|\,\mathrm{d}z\right)\mathrm{d}x\,\mathrm{d}y\right|\\ &\quad\leqslant\left(\int_M\left(\int_{-h}^0|u|\,\mathrm{d}z\right)^4\mathrm{d}x\,\mathrm{d}y\right)^{1/4}|\nabla\partial_z\tilde{T}_1|\\ &\qquad\times\left(\int_M\left(\int_{-h}^0|\chi_m|^2\,\mathrm{d}z\right)^2\mathrm{d}x\,\mathrm{d}y\right)^{1/4}\\ &\quad\leqslant C\|\tilde{T}_1\|(|\chi_m|\|\chi_m\|+|\tilde{\chi}_m|^{1/4}\|\tilde{\chi}_m\|^{3/4}|\chi_m|^{3/4}\|\chi_m\|^{1/4}),\end{aligned} \tag{112}$$

$$\begin{aligned}&\left|\int_\Omega|u||\nabla\chi_m||\tilde{T}_1|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|\\ &\quad\leqslant C\left|\int_M\left(\int_{-h}^0|\nabla\chi|\,\mathrm{d}z\right)\left(\int_{-h}^0|\tilde{T}_1||\nabla\chi_m|\,\mathrm{d}z\right)\mathrm{d}x\,\mathrm{d}y\right|\\ &\quad\leqslant C\int_M\left[\int_{-h}^0|\nabla\chi|\,\mathrm{d}z\max_{-h\leqslant z\leqslant 0}|\tilde{T}_1(\cdot,\cdot,z,t)|\int_{-h}^0|\nabla\chi_m|\,\mathrm{d}z\right]\mathrm{d}x\,\mathrm{d}y\\ &\quad\leqslant C\int_M\left[\int_{-h}^0|\nabla\chi|\,\mathrm{d}z\left(\int_{-h}^0|\tilde{T}_1|^2\,\mathrm{d}z\right)^{1/4}\left(\int_{-h}^0|(\tilde{T}_1)_z|^2\,\mathrm{d}z\right)^{1/4}\right.\\ &\qquad\left.\times\int_{-h}^0|\nabla\chi_m|\,\mathrm{d}z\right]\mathrm{d}x\,\mathrm{d}y\\ &\quad\leqslant C|\tilde{T}_1|^{1/2}\|\tilde{T}_1\|^{1/2}[|\chi_m|^{1/2}\|\chi_m\|^{3/2}+|\tilde{\chi}_m|^{1/4}\|\tilde{\chi}_m\|^{3/4}|\chi_m|^{1/4}\|\chi_m\|^{3/4}]\end{aligned} \tag{113}$$

and

$$\left|\int_\Omega\int_{-h}^0 |u|\,\mathrm{d}z|\nabla\partial_z\chi_m||\tilde T_1|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|$$
$$\leqslant C\int_M\left[\int_{-h}^0|\nabla\chi|\,\mathrm{d}z\left(\int_{-h}^0|\nabla\partial_z\chi_m|^2\,\mathrm{d}z\right)^{1/2}\left(\int_{-h}^0|\tilde T_1|^2\,\mathrm{d}z\right)^{1/2}\right]\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant C|\nabla\partial_z\chi_m|\left(\int_M\left(\int_{-h}^0|\nabla\chi|\,\mathrm{d}z\right)^4\mathrm{d}x\,\mathrm{d}y\right)^{1/4}$$
$$\times\left(\int_M\left(\int_{-h}^0|\tilde T_1|^2\,\mathrm{d}z\right)^2\mathrm{d}x\,\mathrm{d}y\right)^{1/4}$$
$$\leqslant C|\tilde T_1|^{3/4}\|\tilde T_1\|^{1/4}|\chi|^{1/4}\|\chi\|^{3/4}\|\chi_m\|. \tag{114}$$

(iv)

$$\left|\int_\Omega[v''\cdot\nabla\tilde\chi_m+w''\partial_z\tilde\chi_m]\chi_m\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|\leqslant C[\|\tilde\chi_m\||\tilde T_2|^{1/4}\|\tilde T_2\|^{3/4}|\chi_m|^{3/4}\|\chi_m\|^{1/4}$$
$$+\|\tilde\chi_m\||\tilde T_2|^{1/4}\|\tilde T_2\|^{3/4}|\chi_m|^{3/4}\|\chi_m\|^{1/4}$$
$$+|\tilde\chi_m|^{1/2}\|\tilde\chi_m\|^{1/2}|\tilde T_2|^{1/4}\|\tilde T_2\|^{3/4}|\chi_m|^{1/4}\|\chi_m\|^{3/4}$$
$$+|\tilde\chi_m|^{3/4}\|\tilde\chi_m\|^{1/4}|\tilde T_2|^{1/4}\|\tilde T_2\|^{3/4}\|\chi_m\|]. \tag{115}$$

Therefore, by (110)–(115), and Young's inequality, we have

$$\frac{\mathrm{d}|\chi_m|^2}{\mathrm{d}t}\leqslant C\|\tilde\chi_m\|^2+C[1+\|T^*\|^2_{H^2(M)}+\|\tilde T_1\|^2+|\tilde T_1|^6\,\|\tilde T_1\|^2+|\tilde T_2|^{2/3}\|\tilde T_2\|^2]|\chi_m|^2$$
$$+C[1+\|\tilde T_1\|^2+|\tilde T_1|^6\,\|\tilde T_1\|^2+|\tilde T_2|^{2/3}\|\tilde T_2\|^2]|\tilde\chi_m|^2.$$

Thanks to Gronwall inequality, we get

$$|\chi_m(t)|^2\leqslant\left[|\chi_m(t_0)|^2+C\int_{t_0}^t\|\tilde\chi_m(s)\|^2\,\mathrm{d}s+C\left[(1+\max_{t_0\leqslant s\leqslant t}|\tilde T_1(s)|^6)\int_{t_0}^t\|\tilde T_1(s)\|^2\,\mathrm{d}s\right.\right.$$
$$\left.\left.+\max_{t_0\leqslant s\leqslant t}|\tilde T_2(s)|^{2/3}\int_{t_0}^t\|\tilde T_2(s)\|^2\,\mathrm{d}s\right]\max_{t_0\leqslant s\leqslant t}|\tilde\chi_m(s)|^2\right]$$
$$\times\exp\left(C\left[1+\max_{t_0\leqslant s\leqslant t}(|\tilde T_1(s)|^6+|\tilde T_2(s)|^{2/3})\right.\right.$$
$$\left.\left.\times\left(\int_{t_0}^t\|\tilde T_1(s)\|^2\,\mathrm{d}s+\int_{t_0}^t\|\tilde T_2(s)\|^2\,\mathrm{d}s\right)\right]\right)\exp(C(1+\|T^*\|^2_{H^2(\Omega)})(t-t_0)).$$

Note that the right-hand side is bounded, uniformly in $m$, for every fixed $t$. By passing the limit, using the Lebesgue dominant convergence theorem, we obtain

$$|\chi(t)|^2\leqslant|\chi(t_0)|^2\exp\left(C\left[1+\max_{t_0\leqslant s\leqslant t}(|\tilde T_1(s)|^6+|\tilde T_2(s)|^{2/3})\right.\right.$$
$$\left.\left.\times\left(\int_{t_0}^t\|\tilde T_1(s)\|^2\,\mathrm{d}s+\int_{t_0}^t\|\tilde T_2(s)\|^2\,\mathrm{d}s\right)\right]\right)\exp(C(1+\|T^*\|^2_{H^2(\Omega)})(t-t_0)). \tag{116}$$

Recall that $\tilde{T}_1, \tilde{T}_2 \in L^\infty([0, S]; L^2(\Omega)) \cap L^2([0, S]; V_2)$, therefore, (116) implies the continuous dependence of weak solutions on initial data as well as their uniqueness. □

**Corollary 4.** *The weak solution of the system* (52)–(54) *depends continuously on the initial data. That is, the problem is well-posed.*

## 4. Global existence, uniqueness and well-posedness of strong solutions

In the previous section we have proved the existence, uniqueness and well-posedness of the weak solution for the reformulated system (52)–(54). In this section we show the global existence, uniqueness and well-posedness of strong solutions for the system (52)–(54).

**Theorem 5.** *Suppose that $T^* \in H^4(M)$ and $Q \in L^2(\Omega)$. Then for every $T_0(x, y, z) \in V_2$ and $S > 0$, there is a unique strong solution $\tilde{T}$ of the system* (52)–(54)*. Moreover, $\tilde{T}$ satisfies*

$$\|\tilde{T}(t)\|^2 + \int_0^t |\nabla \cdot q(\tilde{T}) - K_v(\tilde{T})_{zz}|^2 \, \mathrm{d}s \leqslant K_s(S, Q, T_0, T^*), \tag{117}$$

*where $0 \leqslant t \leqslant S$ and $K_s(S, Q, T_0, T^*)$ will be specified as in* (133).

**Remark.** The steps of the following proof are formal in the sense that they can be made more rigorous by proving the corresponding estimates first for the Galerkin approximation system based on the eigenfunctions of operator $\nabla \cdot q - K_v(\tilde{T})_{zz}$ with the boundary conditions (53). Then the estimates for the exact solution can be established by passing to the limit in the Galerkin procedure by using the appropriate 'compactness theorems'.

**Proof.** Let $\tilde{T}$ be the weak solution with the initial datum $\tilde{T}_0$. we will show that $\tilde{T}$ is a strong solution if $T_0(x, y, z) \in V_2$. First, we get *a priori* estimate for $|\tilde{T}_z|$. Note that

$$-\int_\Omega \tilde{T}_{zz}\tilde{T} \, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z = \int_\Omega \tilde{T}_z^2 \, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z + \frac{\alpha}{K_v}\int_{z=0} \tilde{T}^2 \, \mathrm{d}x\, \mathrm{d}y$$

and

$$\begin{aligned}
-\int_\Omega & \tilde{T}_{zz}[\nabla \cdot q(\tilde{T}) - K_v(\tilde{T})_{zz}] \, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\
&= \int_\Omega [K_h|\nabla \tilde{T}_z|^2 + K_v|\tilde{T}_{zz}|^2 + \lambda|\nabla \cdot (H^T\nabla \tilde{T}_z)|^2 + \mu|\nabla \tilde{T}_{zz}|^2] \, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z \\
&\quad + \frac{\alpha\lambda}{K_v}\int_{z=0} |\nabla \cdot (H^T\nabla \tilde{T})|^2 \, \mathrm{d}x\, \mathrm{d}y + \frac{\alpha K_h}{K_v}\int_{z=0} |\nabla \tilde{T}|^2 \, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}$$

By taking the inner product of equation (52) with $\tilde{T}_{zz}$ in $L^2(\Omega)$, we reach

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\left(|\tilde{T}_z|^2 + \frac{\alpha}{K_v}\int_{z=0} \tilde{T}^2 \, \mathrm{d}x\, \mathrm{d}y\right) + \frac{\alpha}{K_v}\int_{z=0} [\lambda|\nabla \cdot (H^T\nabla \tilde{T})|^2 + K_h|\nabla \tilde{T}|^2] \, \mathrm{d}x\, \mathrm{d}y \\
&+ K_h|\nabla \tilde{T}_z|^2 + K_v|\tilde{T}_{zz}|^2 + \lambda|\nabla \cdot (H^T\nabla \tilde{T}_z)|^2 + \mu|\nabla \tilde{T}_{zz}|^2 \\
&= \int_\Omega [Q^* - v \cdot \nabla \tilde{T} - w\partial_z \tilde{T} - v \cdot \nabla T^*]\tilde{T}_{zz} \, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z.
\end{aligned}$$

Let us consider the above equation term by term

(i) It is clear that

$$\left|\int_\Omega Q^*\tilde{T}_{zz}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| = \left|\int_\Omega (Q-\nabla\cdot q(T^*))\tilde{T}_{zz}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| \leqslant C(|Q| + \|T^*\|_{H^4(M)})|\tilde{T}_{zz}|. \tag{118}$$

(ii) An integration by parts yields

$$\int_\Omega [-v\cdot\nabla\tilde{T} - w\tilde{T}_z - v\cdot\nabla T^*]\tilde{T}_{zz}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \int_\Omega [v_z\cdot\nabla\tilde{T} + w_z\partial_z\tilde{T} + v_z\cdot\nabla T^* + v\cdot\nabla\tilde{T}_z + w\tilde{T}_{zz}]\tilde{T}_z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$+\frac{\alpha}{K_v}\int_{z=0}[v\cdot\nabla(\tilde{T}+T^*)]\tilde{T}\,\mathrm{d}x\,\mathrm{d}y.$$

Moreover, by (36) and the boundary conditions (38)–(40), we have

$$\int_\Omega (v\cdot\nabla\tilde{T}_z + w\tilde{T}_{zz})\tilde{T}_z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0. \tag{119}$$

Using (58)–(61) and the boundary conditions (53), we obtain

$$\int_\Omega [v_z\cdot\nabla\tilde{T} + w_z\tilde{T}_z + v_z\cdot\nabla T^*]\tilde{T}_z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \frac{\alpha}{K_v}\int_{z=0}[v\cdot\nabla(\tilde{T}+T^*)]\tilde{T}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \int_\Omega \frac{\epsilon}{\epsilon^2+f^2}|\nabla\tilde{T}+\nabla T^*|^2\tilde{T}_z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \int_\Omega \nabla\cdot v|\partial_z\tilde{T}|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$+\frac{\alpha}{K_v}\int_{z=0}\left[\int_{-h}^0 \frac{\epsilon}{\epsilon^2+f^2}\left(1+\frac{\xi}{h}\right)|\nabla\tilde{T}(x,y,\xi,t)\right.$$
$$\left.+\nabla T^*(x,y)|^2\,\mathrm{d}\xi\right]\tilde{T}\,\mathrm{d}x\,\mathrm{d}y. \tag{120}$$

Thanks to (73), we have

$$\left|\int_\Omega \frac{\epsilon}{\epsilon^2+f^2}|\nabla\tilde{T}+\nabla T^*|^2\tilde{T}_z\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| \leqslant C(\|\nabla\tilde{T}\|^2_{L^4(\Omega)} + \|\nabla T^*\|^2_{L^4(M)})|\tilde{T}_z|$$
$$\leqslant C(|\Delta\tilde{T}|^2 + \|T^*\|^2_{H^4(M)})|\tilde{T}_z|$$
$$\leqslant C(\|\tilde{T}\|^2 + \|T^*\|^2_{H^4(M)})|\tilde{T}_z|. \tag{121}$$

Similarly,

$$\left|\frac{\alpha}{K_v}\int_{z=0}\left[\int_{-h}^0 \frac{\epsilon}{\epsilon^2+f^2}\left(1+\frac{\xi}{h}\right)|\nabla\tilde{T}(x,y,\xi,t)+\nabla T^*(x,y)|^2\,\mathrm{d}\xi\right]\tilde{T}\,\mathrm{d}x\,\mathrm{d}y\right|$$
$$\leqslant C(\|\nabla\tilde{T}\|^2_{L^4(\Omega)} + \|\nabla T^*\|^2_{L^4(M)})\|\tilde{T}(z=0)\|_{L^2(M)}$$
$$\leqslant C(|\Delta\tilde{T}|^2 + \|T^*\|^2_{H^4(M)})\|\tilde{T}(z=0)\|_{L^2(M)}$$
$$\leqslant C(\|\tilde{T}\|^2 + \|T^*\|^2_{H^4(M)})\|\tilde{T}(z=0)\|_{L^2(M)}. \tag{122}$$

Following the steps to get the estimate (113) we obtain

$$\left|\int_\Omega \nabla\cdot v|\partial_z\tilde{T}|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| \leqslant C(|\Delta\tilde{T}| + \|\Delta T^*\|_{L^2(M)})\|\tilde{T}_z\|_{L^2(M)}\|\nabla\tilde{T}_z\|_{L^2(M)}$$
$$\leqslant C(\|\tilde{T}\| + \|T^*\|_{H^4(M)})\|\tilde{T}_z\|_{L^2(M)}\|\nabla\tilde{T}_z\|_{L^2(M)}. \tag{123}$$

Therefore, from (118)–(123), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\tilde{T}_z|^2 + \frac{\alpha}{K_v}\int_{z=0}\tilde{T}^2\,\mathrm{d}x\,\mathrm{d}y\right) + K_h|\nabla\tilde{T}_z|^2 + K_v|\tilde{T}_{zz}|^2 + \lambda|\nabla\cdot(H^T\nabla\tilde{T}_z)|^2 + \mu|\nabla\tilde{T}_{zz}|^2$$
$$+\frac{\alpha}{K_v}\int_{z=0}[\lambda|\nabla\cdot(H^T\nabla\tilde{T})|^2 + K_h|\nabla\tilde{T}|^2]\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant C(|Q| + \|T^*\|_{H^4(M)})|\tilde{T}_{zz}| + C(\|\tilde{T}\|^2 + \|T^*\|^2_{H^4(M)})|\tilde{T}_z|$$
$$+C(\|\tilde{T}\|^2 + \|T^*\|^2_{H^4(M)})\|\tilde{T}(z=0)\|_{L^2(M)}$$
$$+C(\|\tilde{T}\| + \|T^*\|_{H^4(M)})\|\tilde{T}_z\|_{L^2(M)}\|\nabla\tilde{T}_z\|_{L^2(M)}.$$

By Cauchy–Schwartz inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(|\tilde{T}_z|^2 + \frac{\alpha}{K_v}\int_{z=0}\tilde{T}^2\,\mathrm{d}x\,\mathrm{d}y\right) + K_h|\nabla\tilde{T}_z|^2 + K_v|\tilde{T}_{zz}|^2 + \lambda|\nabla\cdot(H^T\nabla\tilde{T}_z)|^2 + \mu|\nabla\tilde{T}_{zz}|^2$$
$$+\frac{\alpha}{K_v}\int_{z=0}[\lambda|\nabla\cdot(H^T\nabla\tilde{T})|^2 + K_h|\nabla\tilde{T}|^2]\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant C(|Q|^2 + \|T^*\|^2_{H^4(M)} + \|\tilde{T}\|^2)$$
$$+C(\|\tilde{T}\|^2 + \|T^*\|^2_{H^4(M)})(\|\tilde{T}(z=0)\|^2_{L^2(M)} + |\tilde{T}_z|^2). \tag{124}$$

Again, by Gronwall inequality, we get

$$|\tilde{T}_z(t)|^2 + \frac{\alpha}{K_v}\int_{z=0}\tilde{T}^2(t)\,\mathrm{d}x\,\mathrm{d}y \leqslant K_z(t, Q, \tilde{T}_0, T^*), \tag{125}$$

where

$$K_z(t, Q, \tilde{T}_0, T^*) = \exp(C(1 + \|T^*\|^2_{H^2(M)})t + K_1(t, Q, \tilde{T}_0, T^*))$$
$$\times[\|T_0\|^2 + \|T^*\|^2_{H^2(M)} + K_1(t, Q, \tilde{T}_0, T^*) + (|Q|^2 + \|T^*\|^2_{H^4(M)})t] \tag{126}$$

and $K_1(t, Q, \tilde{T}_0, T^*)$ is as in (90). Finally, let us show that

$$\tilde{T}(x, y, z, t) \in L^\infty([0, S], V_2) \cap L^2([0, S], V_4).$$

We take the inner product of equation (52) with $\nabla\cdot q(\tilde{T}) - K_v\tilde{T}_{zz}$ in $L^2(\Omega)$ to get

$$\frac{1}{2}\frac{\mathrm{d}\|\tilde{T}\|^2}{\mathrm{d}t} + |\nabla\cdot q(\tilde{T}) - K_v\tilde{T}_{zz}|^2$$
$$= \int_\Omega[Q^* - v\cdot\nabla\tilde{T} - w\partial_z\tilde{T} - v\cdot\nabla T^*][\nabla\cdot q(\tilde{T}) - K_v\tilde{T}_{zz}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Note that

(i)

$$\left|\int_\Omega Q^*[\nabla\cdot q(\tilde{T}) - K_v\tilde{T}_{zz}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| \leqslant |Q^*||\nabla\cdot q(\tilde{T}) - K_v\tilde{T}_{zz}|$$
$$\leqslant [|Q| + C\|T^*\|_{H^4(M)}]|\nabla\cdot q(\tilde{T}) - K_v\tilde{T}_{zz}|. \tag{127}$$

(ii)

$$\left|\int_\Omega v\cdot\nabla T^*[\nabla\cdot q(\tilde{T}) - K_v\tilde{T}_{zz}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| \leqslant C|v|\|\nabla T^*\|_{L^\infty(M)}|\nabla\cdot q(\tilde{T}) - K_v\tilde{T}_{zz}|$$
$$\leqslant C\|\tilde{T}\|\|T^*\|_{H^4(M)}|\nabla\cdot q(\tilde{T}) - K_v\tilde{T}_{zz}|. \tag{128}$$

(iii) Following similar steps to those which led to the estimate (113) we have

$$\left|\int_\Omega v\cdot\nabla\tilde{T}[\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|$$
$$\leqslant C(|\nabla\tilde{T}|+\|\nabla T^*\|_{L^2(M)})(\|\Delta\tilde{T}\|_{L^2(\Omega)}+\|T^*\|_{H^2(M)})|\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}|$$
$$\leqslant C(\|T^*\|^2_{H^2(M)}+\|\tilde{T}\|^2)|\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}|. \tag{129}$$

(iv) Next, let us deal with the last term

$$\left|\int_\Omega w\partial_z\tilde{T}[\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|\leqslant C\|w\|_{L^\infty(\Omega)}|\partial_z\tilde{T}||\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}|.$$

Note that from (61)

$$\|w\|_{L^\infty(\Omega)}\leqslant C\int_{-h}^0[\|\Delta\tilde{T}(z,t)\|_{L^\infty(M)}+\|\Delta T^*\|_{L^\infty(M)}]\,\mathrm{d}z.$$

Hence

$$\|w\|_{L^\infty(\Omega)}\leqslant C\left(\int_{-h}^0\|\Delta\tilde{T}(z,t)\|^{1/2}_{L^2(M)}\|\Delta\tilde{T}(z,t)\|^{1/2}_{H^2(M)}\,\mathrm{d}z+\|\Delta T^*\|_{L^\infty(M)}\right).$$

By Cauchy–Schwarz inequality, (77) and proposition 2 we get

$$\|w\|_{L^\infty(\Omega)}\leqslant C(\|\Delta T^*\|_{L^\infty(M)}+\|\tilde{T}\|^{1/2}|\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}|^{1/2}).$$

Hence

$$\left|\int_\Omega w\partial_z\tilde{T}[\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}]\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right|\leqslant C[\|T^*\|_{H^4(M)}+\|\tilde{T}\|^{1/2}|\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}|^{1/2}]$$
$$\times|\partial_z\tilde{T}||\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}|. \tag{130}$$

Therefore, from the estimates (127)–(130) we have

$$\frac{1}{2}\frac{\mathrm{d}\|\tilde{T}\|^2}{\mathrm{d}t}+|\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}|^2\leqslant C|\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}|[1+|Q|+\|T^*\|^2_{H^4(M)}$$
$$+\|\tilde{T}\|^2+|\partial_z\tilde{T}|^2]+C\|\tilde{T}\|^{1/2}|\partial_z\tilde{T}||\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}|^{3/2}.$$

By using Young's inequality we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{T}\|^2+|\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}|^2$$
$$\leqslant C[1+|Q|^2+\|T^*\|^4_{H^4(M)}+|\tilde{T}_z|^4]+C[1+\|\tilde{T}\|^2+|\tilde{T}_z|^4]\|\tilde{T}\|^2. \tag{131}$$

Again, by Gronwall inequality we conclude

$$\|\tilde{T}(t)\|^2+\int_0^t|\nabla\cdot q(\tilde{T})-K_v\tilde{T}_{zz}|^2\,\mathrm{d}s\leqslant K_s(t,Q,\tilde{T}_0,T^*), \tag{132}$$

where

$$K_s(S,Q,\tilde{T}_0,T^*)=\exp(C[1+(K_z(t,Q,\tilde{T}_0,T^*))^2]t+K_1(t,Q,\tilde{T}_0,T^*))$$
$$\times[\|\tilde{T}_0\|^2+|Q|^2+\|T^*\|^4_{H^4(M)}+K_z(t,Q,\tilde{T}_0,T^*)^2], \tag{133}$$

where $K_1(t,Q,\tilde{T}_0,T^*)$ and $K_z(t,Q,\tilde{T}_0,T^*)$ are as in (90) and (126), respectively. Since the strong solution is a weak solution, by theorem 3, the strong solution is unique. □

## 5. The global attractor

In previous sections we have proved the existence and uniqueness of the weak and strong solution of the system (52)–(54). In this section we show the existence of the global attractor. Moreover, we give an upper bound, which are not necessarily optimal, for the dimension of the global attractor. Denote by $\tilde{T}(t) = S(t)\tilde{T}_0$ the solution of the system (52)–(54) with initial data $\tilde{T}_0$. As a result of theorems 3 and 5, one can show that

$$\tilde{T}(t) = S(t)\tilde{T}_0 \in L^2(\Omega) \qquad \text{for all } \tilde{T}_0 \in L^2(\Omega), \quad t \geqslant 0$$

and

$$\tilde{T}(t) = S(t)\tilde{T}_0 \in V_2 \qquad \text{for all } \tilde{T}_0 \in V_2, \quad t \geqslant 0.$$

Since, in this section, we only consider the long-time behaviour of solutions of the system (52)–(54), by theorems 3 and 5, we conclude that $\tilde{T}(t) \in L^\infty_{\text{loc}}((0, S], V_2)$ for every $\tilde{T}(0) \in L^2(\Omega)$. As a result, one can easily show that

$$\partial_t \tilde{T} \in L^2_{\text{loc}}((0, S], V_2'). \tag{134}$$

**Theorem 6.** *Suppose that $Q \in L^2(\Omega)$ and $T^* \in H^4(M)$. Then, there is a global attractor $\mathcal{A} \subset L^2(\Omega)$ for the system* (52)–(54). *Moreover, $\mathcal{A}$ has finite Hausdorff and fractal dimensions.*

**Proof.** First, let us show that there are absorbing balls in $L^2(\Omega)$ and $V_2$. Let $\tilde{T}$ be the solution of the system (52)–(54) with initial datum $\tilde{T}_0 \in L^2(\Omega)$ and $|T_0| = |\tilde{T}_0 + T^*| \leqslant \rho$. By theorems 3 and 5, there is a $t_0$ such that

$$|T(t_0)| = |\tilde{T}(t_0) + T^*| \leqslant 2\rho.$$

From now on we assume that $t \geqslant t_0$. By taking the $H'$ dual action to equation (53) with $\tilde{T}$, we obtain

$$\langle \partial_t \tilde{T} + \nabla \cdot q(\tilde{T}) - K_v \tilde{T}_{zz}, \tilde{T} \rangle + \langle v \cdot \nabla \tilde{T} + w \partial_z \tilde{T} + v \cdot \nabla T^*, \tilde{T} \rangle = \langle Q^*, \tilde{T} \rangle.$$

Applying (134) and Lions' lemma (cf [26] lemma 1.2, p 260), we reach

$$\langle \partial_t \tilde{T}, \tilde{T} \rangle = \frac{1}{2} \frac{\mathrm{d}|\tilde{T}|^2}{\mathrm{d}t}.$$

Moreover, we have

$$\langle \nabla \cdot q(\tilde{T}) - K_v \tilde{T}_{zz}, \tilde{T} \rangle = \|\tilde{T}\|^2.$$

In addition,

$$\langle v \cdot \nabla \tilde{T} + w \partial_z \tilde{T} + v \cdot \nabla T^*, \tilde{T} \rangle = \int_\Omega [v \cdot \nabla \tilde{T} + w \partial_z \tilde{T} + v \cdot \nabla T^*] \tilde{T} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

and

$$\langle Q^*, \tilde{T} \rangle = \int_\Omega Q^* \, \tilde{T} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z,$$

as long as the integrals on the right-hand side make sense. Therefore, we have

$$\frac{1}{2} \frac{\mathrm{d}|\tilde{T}|^2}{\mathrm{d}t} + \|\tilde{T}\|^2 + \int_\Omega [v \cdot \nabla \tilde{T} + w \partial_z \tilde{T} + v \cdot \nabla T^*] \tilde{T} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \int_\Omega Q^* \, \tilde{T} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{135}$$

By taking $\psi = T^*$ in the weak formulation (70), we get

$$\int_\Omega \tilde{T}(t)T^* \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - \int_\Omega \tilde{T}(t_0)\, T^* \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \int_{t_0}^t a(\tilde{T}, T^*)$$
$$+ \int_{t_0}^t \int_\Omega [v \cdot \nabla\tilde{T} + w\partial_z\tilde{T} + v \cdot \nabla T^*]\, T^* \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_{t_0}^t \int_\Omega Q^*\, T^* \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

It is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \tilde{T}(t)T^* \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + a(\tilde{T}, T^*)$$
$$+ \int_\Omega [v \cdot \nabla\tilde{T} + w\partial_z\tilde{T} + v \cdot \nabla T^*]\, T^* \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_\Omega Q^*\, T^* \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \qquad (136)$$

By adding (135) and (136), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\tilde{T}|^2 + 2\int_\Omega \tilde{T}(t)T^* \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right) + \|\tilde{T}\|^2 + a(\tilde{T}, T^*)$$
$$+ \int_\Omega [v \cdot \nabla\tilde{T} + w\partial_z\tilde{T} + v \cdot \nabla T^*](\tilde{T} + T^*) \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \int_\Omega Q^*(\tilde{T} + T^*) \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Note that

(i)
$$|\tilde{T}|^2 + \int_\Omega 2\tilde{T}\, T^* \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |T|^2 - \int_\Omega |T^*|^2 \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z;$$

(ii)
$$\int_\Omega [v \cdot \nabla\tilde{T} + w\partial_z\tilde{T} + v \cdot \nabla T^*](\tilde{T} + T^*) \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_\Omega [v \cdot \nabla T + w\partial_z T\,]T \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z;$$

(iii)
$$\int_\Omega Q^*\,(\tilde{T} + T^*) \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_\Omega Q\,T \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z - a(\tilde{T}, T^*) - \|T^*\|^2$$
$$+\alpha \int_{z=0}\left[T\,T^* + \frac{\mu}{K_v}\nabla T \cdot \nabla T^*\right] \mathrm{d}x\,\mathrm{d}y.$$

Therefore, we get

$$\frac{1}{2}\frac{\mathrm{d}|T|^2}{\mathrm{d}t} + \|T\|^2 + \int_\Omega [v \cdot \nabla T + w\partial_z T]\, T \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$
$$= \int_\Omega Q\,T \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z + \alpha\int_{z=0}\left[T\,T^* + \frac{\mu}{K_v}\nabla T \cdot \nabla T^*\right] \mathrm{d}x\,\mathrm{d}y.$$

By integration by parts and (40), we obtain

$$\int_\Omega [v \cdot \nabla T + w\partial_z T] \, \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = 0. \qquad (137)$$

By applying Cauchy–Schwarz inequality, we reach

$$\left|\int_{z=0}\left[T\,T^* + \frac{\mu}{K_v}\nabla T \cdot \nabla T^*\right] \mathrm{d}x\,\mathrm{d}y\right| \leqslant \|T^*\|_{H^1(M)}\left[\|T\|_{L^2(M)} + \frac{\mu}{K_v}\|\nabla T\|_{L^2(M)}\right]$$
$$\leqslant C_2^{1/2}\|T^*\|_{H^1(M)}\left[\|T\|_{H^1(\Omega)} + \frac{\mu}{K_v}\|\nabla T\|_{H^1(\Omega)}\right]$$
$$\leqslant C_2^{1/2}\left(1 + \frac{\mu}{K_v}\right)\|T^*\|_{H^1(M)}\|T\|. \qquad (138)$$

It is clear that

$$\left| \int_\Omega Q\, T \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \right| \leqslant |Q||T|. \tag{139}$$

Therefore, by the above estimates (137)–(139), we obtain

$$\frac{\mathrm{d}|T|^2}{\mathrm{d}t} + \|T\|^2 \leqslant C_2\alpha^2 \left(1 + \frac{\mu}{K_v}\right)^2 \|T^*\|^2_{H^1(M)} + 2|Q||T|.$$

By Cauchy–Schwarz inequality and (74), we have

$$\frac{\mathrm{d}|T|^2}{\mathrm{d}t} + \frac{1}{2}\|T\|^2 \leqslant C_2\alpha^2 \left(1 + \frac{\mu}{K_v}\right)^2 \|T^*\|^2_{H^1(M)} + 2C_0|Q|^2, \tag{140}$$

where $C_0$ is as in (74). Thus, again, by (74), we obtain

$$\frac{\mathrm{d}|T|^2}{\mathrm{d}t} + \frac{1}{2C_0}|T|^2 \leqslant C_2\alpha^2 \left(1 + \frac{\mu}{K_v}\right)^2 \|T^*\|^2_{H^1(M)} + 2C_0|Q|^2.$$

By Gronwall's lemma, we get

$$|T(t)|^2 \leqslant |T(t_0)|^2 \exp\left(-\frac{1}{2C_0}\,t\right) + 2C_0C_2\alpha^2 \left(1 + \frac{\mu}{K_v}\right)^2 \|T^*\|^2_{H^1(M)} + 4C_0^2|Q|^2.$$

As a result of the above, when $t$ is large enough such that

$$|T(t_0)|^2 \exp\left(-\frac{1}{2C_0}\,t\right) \leqslant 2C_0C_2\alpha^2 \left(1 + \frac{\mu}{K_v}\right)^2 \|T^*\|^2_{H^1(M)} + 4C_0^2|Q|^2,$$

we have

$$|T(t)|^2 \leqslant \tilde{R}_a(T^*, Q), \tag{141}$$

where

$$\tilde{R}_a(T^*, Q) = 4C_0C_2\alpha^2 \left(1 + \frac{\mu}{K_v}\right)^2 \|T^*\|^2_{H^1(M)} + 8C_0^2|Q|^2. \tag{142}$$

In particular,

$$\limsup_{t\to\infty} |T(t)|^2 \leqslant 2C_0C_2\alpha^2 \left(1 + \frac{\mu}{K_v}\right)^2 \|T^*\|^2_{H^1(M)} + 4C_0^2|Q|^2.$$

In other words, when $t$ is large enough we have

$$|\tilde{T}(t)|^2 = |T(t) - T^*|^2 \leqslant R_a(T^*, Q), \tag{143}$$

where

$$R_a(T^*, Q) = 2\tilde{R}_a(T^*, Q) + 2\|T^*\|^2_{L^2(M)}, \tag{144}$$

where $\tilde{R}_a(T^*, Q)$ is as in (142). Therefore, there is an absorbing ball in $L^2(\Omega)$ with radius $R_a(T^*, Q)$ for system (52)–(54).

Next, we show that there is an absorbing ball in $V_2$. First, note that from (140), we have

$$\int_t^{t+r} \|T(s)\|^2 \, \mathrm{d}s \leqslant 2|T(t)| + \left[4C_0C_2\alpha^2 \left(1 + \frac{\mu}{K_v}\right)^2 \|T^*\|^2_{H^1(M)} + 8C_0^2|Q|^2\right] r.$$

Therefore, by (143), when $t$ is large enough, we get

$$\int_t^{t+r} \|T(s)\|^2 \, \mathrm{d}s \leqslant K_r(r, Q, T^*), \tag{145}$$

where

$$K_r(r, Q, T^*) = 2R_a(T^*, Q) + \left[4C_0C_2\alpha^2\left(1+\frac{\mu}{K_v}\right)^2\|T^*\|^2_{H^1(M)} + 8C_0^2|Q|^2\right]r \tag{146}$$

and $R_a(T^*, Q)$ is as in (144).

From the proof of theorem 5 we recall the inequality (124):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\tilde{T}_z|^2 + \frac{\alpha}{K_v}\int_{z=0}\tilde{T}^2\,\mathrm{d}x\,\mathrm{d}y\right) &+ K_h|\nabla\tilde{T}_z|^2 + K_v|\tilde{T}_{zz}|^2 + \lambda|\nabla\cdot(H^T\nabla\tilde{T}_z)|^2 + \mu|\nabla\tilde{T}_{zz}|^2\\ &+\frac{\alpha}{K_v}\int_{z=0}[\lambda|\nabla\cdot(H^T\nabla\tilde{T})|^2 + K_h|\nabla\tilde{T}|^2]\,\mathrm{d}x\,\mathrm{d}y\\ &\leqslant C(|Q|^2 + \|T^*\|^2_{H^4(M)} + \|\tilde{T}\|^2)\\ &\quad+C(\|\tilde{T}\|^2 + \|T^*\|^2_{H^4(M)})(\|\tilde{T}(z=0)\|^2_{L^2(M)} + |\tilde{T}_z|^2).\end{aligned}$$

Thus,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(|\tilde{T}_z|^2 + \frac{\alpha}{K_v}\int_{z=0}\tilde{T}^2\,\mathrm{d}x\,\mathrm{d}y\right) &\leqslant C(|Q|^2 + \|T^*\|^2_{H^4(M)} + \|\tilde{T}\|^2)\\ &+C(\|\tilde{T}\|^2 + \|T^*\|^2_{H^4(M)})\left(\frac{\alpha}{K_v}\|\tilde{T}(z=0)\|^2_{L^2(M)} + |\tilde{T}_z|^2\right).\end{aligned}$$

By applying the uniform Gronwall inequality (cf, e.g., [27], p 89) and (145), we obtain, when $t$ is large enough

$$|\tilde{T}_z(t)|^2 + \frac{\alpha}{K_v}\|\tilde{T}(z=0)\|^2_{L^2(M)} \leqslant R_z(r, T^*, Q), \tag{147}$$

where $r > 0$ is fixed and

$$\begin{aligned}R_z(r, T^*, Q) = C&\left[\frac{K_r(r, T^*, Q)}{r} + \|T^*\|^2_{H^4(M)}r + |Q|^2r + K_r(r, T^*, Q)\right]\\ &\times\exp(C[K_r(r, T^*, Q) + \|T^*\|^2_{H^4(M)}r]).\end{aligned} \tag{148}$$

Let us recall the inequality (131):

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{T}\|^2 + |\nabla\cdot q(\tilde{T}) - K_v\tilde{T}_{zz}|^2\\ &\qquad\leqslant C[1 + |Q|^2 + \|T^*\|^4_{H^4(M)} + |\tilde{T}_z|^4] + C[1 + \|\tilde{T}\|^2 + |\tilde{T}_z|^4]\|\tilde{T}\|^2.\end{aligned}$$

Thus,

$$\frac{\mathrm{d}\|\tilde{T}\|^2}{\mathrm{d}t} \leqslant C[1 + |Q|^2 + \|T^*\|^4_{H^4(M)} + |\tilde{T}_z|^4] + C[1 + \|\tilde{T}\|^2 + |\tilde{T}_z|^4]\|\tilde{T}\|^2.$$

Thanks to the uniform Gronwall inequalities (145) and (147), we obtain, when $t$ is large enough

$$\|\tilde{T}(t)\| \leqslant R_v(r, T^*, Q), \tag{149}$$

where $r > 0$ is fixed and

$$\begin{aligned}(R_v(r, T^*, Q))^2 = C&\left[\frac{K_r(r, Q, T^*)}{r} + r + |Q|^2r + \|T^*\|^4_{H^4(M)}r + R_z(r, Q, T^*)^2r\right]\\ &\times\exp(C[r + R_z(r, Q, T^*)^2r + K_r(r, Q, T^*)]).\end{aligned} \tag{150}$$

Therefore, we have shown that there is an absorbing ball $\mathcal{B}$ in $V_2$ with radius $R_v(r, T^*, Q)$. From the proofs of theorems 3 and 5 we conclude that the operator $S(t)$ is a compact operator.

Following the standard procedure (cf, e.g. [5–7, 10, 27] for details), one can prove that there is a global attractor

$$\mathcal{A} = \cap_{t>0} S(t)\mathcal{B} \subset V_2.$$

Moreover, $\mathcal{A}$ is compact in $L^2(\Omega)$ due to the compact embedding of $V_2$ in $L^2(\Omega)$.

In addition to the compactness of the semi-group $S(t)$ one can show its differentiability on $\mathcal{A}$ with respect to the initial data. Therefore, one can use the formula (cf [5, 6, 27]) to get an upper bound for the dimension of the global attractor $\mathcal{A}$. Let $\tilde{T}$ be a given solution of the system (52)–(54) with $\tilde{T} \in \mathcal{A}$. Since it is on the global attractor $\mathcal{A}$, $\tilde{T}$ is a strong solution to the system (52)–(54). It is clear that the first variation equations of the system (52)–(54) around $\tilde{T}$ read:

$$\partial_t \chi = F'(\tilde{T})\chi, \tag{151}$$

$$\left(\partial_z \chi + \frac{\alpha}{K_v}\chi\right)\bigg|_{z=0} = 0, \qquad \partial_z \chi|_{z=-h} = 0, \qquad \frac{\partial \chi}{\partial \vec{e}}\bigg|_{\Gamma_s} = 0, \qquad q(\chi)\cdot\vec{n}|_{\Gamma_s} = 0, \tag{152}$$

$$\chi(x, y, z, 0) = \zeta, \tag{153}$$

where $\chi$ are the unknown perturbations about $\tilde{T}$ with a given initial perturbation $\zeta \in L^2(\Omega)$. Moreover, here

$$F'(\tilde{T})\chi = -[\nabla\cdot q(\chi) - K_v\chi_{zz} + u\cdot\nabla(\tilde{T} + T^*) + v\cdot\nabla\chi - \tilde{w}\partial_z\tilde{T} + w\partial_z\chi],$$

where $v = (v_1, v_2)$, $w$ are as in (58)–(61), and

$$\begin{aligned} u_1 = &\int_{-h}^{z} \frac{\epsilon\chi_x(x, y, \xi, t) + f\chi_y(x, y, \xi, t)}{\epsilon^2 + f^2}\,\mathrm{d}\xi \\ &- \frac{1}{h}\int_{-h}^{0}\int_{-h}^{z} \frac{\epsilon\chi_x(x, y, \xi, t) + f\chi_y(x, y, \xi, t)}{\epsilon^2 + f^2}\,\mathrm{d}\xi\,\mathrm{d}z, \end{aligned} \tag{154}$$

$$\begin{aligned} u_2 = &\int_{-h}^{z} \frac{-f\chi_x(x, y, \xi, t) + \epsilon\chi_y(x, y, \xi, t)}{\epsilon^2 + f^2}\,\mathrm{d}\xi \\ &- \frac{1}{h}\int_{-h}^{0}\int_{-h}^{z} \frac{-f\chi_x(x, y, \xi, t) + \epsilon\chi_y(x, y, \xi, t)}{\epsilon^2 + f^2}\,\mathrm{d}\xi\,\mathrm{d}z, \end{aligned} \tag{155}$$

$$\begin{aligned} \tilde{w} = -&\int_{-h}^{z}\int_{-h}^{\eta}\left[\frac{\epsilon\Delta\chi(x, y, \xi, t) - f_0\chi_x(x, y, \xi, t)}{\epsilon^2 + f^2}\right. \\ &\left. + \frac{2f_0 f}{(\epsilon^2 + f^2)^2}(-f\chi_x(x, y, \xi, t) + \epsilon\chi_y(x, y, \xi, t))\right]\mathrm{d}\xi\,\mathrm{d}\eta \\ &+ \frac{z+h}{h}\int_{-h}^{0}\int_{-h}^{z}\left[\frac{\epsilon\Delta\chi(x, y, \xi, t) - f_0\chi_x(x, y, \xi, t)}{\epsilon^2 + f^2}\right. \\ &\left. - \frac{2f_0 f}{(\epsilon^2 + f^2)^2}(-f\chi_x(x, y, \xi, t) + \epsilon\chi_y(x, y, \xi, t))\right]\mathrm{d}\xi\,\mathrm{d}z. \end{aligned} \tag{156}$$

It is not difficult to show that the above, coupled second-order elliptic and linear parabolic, system has a unique solution $\chi(t)$. Moreover, for $t > 0$,

$$\chi(t) \in V_2.$$

For any positive integer $m$ we consider the volume element $|\chi_1(t) \wedge \cdots \wedge \chi_m(t)|_{\wedge^m L^2(\Omega)}$, we have the following trace formula (cf [5, 6, 27])

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\chi_1(t) \wedge \cdots \wedge \chi_m(t)|^2_{\wedge^m L^2(\Omega)} = \mathrm{Tr}(\tilde{P}_m(t) \circ F'(\tilde{T}(t)) \circ \tilde{P}_m(t))|\chi_1(t) \wedge \cdots \wedge \chi_m(t)|^2_{\wedge^m L^2(\Omega)},$$

which gives

$$|\chi_1(t) \wedge \cdots \wedge \chi_m(t)|^2_{\wedge^m L^2(\Omega)} = |\zeta_1 \wedge \cdots \wedge \zeta_m|^2_{\wedge^m L^2(\Omega)} \exp \int_0^t \mathrm{Tr}(\tilde{P}_m(s) \circ F'(\tilde{T}(s)) \circ \tilde{P}_m(s))\,\mathrm{d}s, \qquad (157)$$

where $\chi_1(s), \ldots, \chi_m(s)$ are the solutions of (151)–(156) corresponding to the initial data $\zeta_1, \ldots, \zeta_m$, respectively. $\mathrm{Tr}(\tilde{P}_m(s) \circ F'(\tilde{T}(s)) \circ \tilde{P}_m(s))$ is the trace of the linear operator $(\tilde{P}_m(s) \circ F'(\tilde{T}(s)) \circ \tilde{P}_m(s))$ and $\tilde{P}_m(s)$ is the $L^2(\Omega)$ orthogonal projector onto the space spanned by $\{\chi_1(s), \ldots, \chi_m(s)\}$. Note that $\{\chi_1(s), \ldots, \chi_m(s)\}$ are linearly independent for every $s \geqslant 0$ if and only if $\{\zeta_1, \ldots, \zeta_m\}$ are linearly independent. Hence, from now on we assume that $\{\zeta_1, \ldots, \zeta_m\}$ are linearly independent. Let $\{\psi_1(s), \ldots, \psi_m(s)\}$ be an $L^2(\Omega)$ orthonormal basis of the space spanned by $\{\chi_1(s), \ldots, \chi_m(s)\}$. Note that $\{\psi_1(s), \ldots, \psi_m(s)\}$ are in $V_2$ for $s > 0$. Thus, we have

$$\mathrm{Tr}(\tilde{P}_m(s) \circ F'(\tilde{T}(s)) \circ \tilde{P}_m(s)) = \sum_{j=1}^{m} (F'(\tilde{T}(s))\psi_j(s), \psi_j(s)).$$

Observe that

$$(F'(\tilde{T}(s))\psi_j(s), \psi_j(s)) = -\|\psi_j(s)\|^2 + \int_\Omega [u_j \cdot \nabla(\tilde{T} + T^*) + \tilde{w}_j \partial_z \tilde{T}]\psi_j(s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

where, for $j = 1, 2, \ldots, m$, $u_j(x, y, z, s)$ and $\tilde{w}_j(x, y, z, s)$ are as in (154)–(156), respectively, but with $\psi_j(s)$ replacing $\chi$. Following the same steps that led the estimates (113), we have

$$\left|\int_\Omega u_j \cdot \nabla(\tilde{T} + T^*)\psi_j(s)\right| \leqslant C(\|\tilde{T}\| + \|T^*\|_{H^1(M)})|\psi_j(s)|\|\psi_j(s)\|$$

and

$$\|\tilde{w}_j \partial_z \tilde{T}\psi_j(s)| \leqslant C|\tilde{T}|^{3/4}\|\nabla\tilde{T}\|^{1/4}|\psi_j(s)|^{1/4}\|\psi_j(s)\|^{7/4}.$$

Recall that $|\psi_j| = 1$, for $j = 1, 2, \ldots, m$. Thus,

$$\left|\int_\Omega [u_j \cdot \nabla(\tilde{T} + T^*) + \tilde{w}\partial_z \tilde{T}]\psi_j(s)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\right| \leqslant C(\|\tilde{T}\| + \|T^*\|_{H^1(M)})\|\psi_j(s)\| + C|\tilde{T}|^{3/4}\|\nabla\tilde{T}\|^{1/4}\|\psi_j(s)\|^{7/4}.$$

By using Young's inequality and the above estimate, we have

$$(F'(\tilde{T}(s))\psi_j(s), \psi_j(s)) \leqslant -\tfrac{1}{2}\|\psi_j(s)\|^2 + C[\|\tilde{T}(s)\|^2 + \|T^*\|^2_{H^1(M)} + |\tilde{T}|^6\|\nabla\tilde{T}\|^2].$$

By (79), we have

$$\sum_{j=1}^{m} \|\psi_j(s)\|^2 \geqslant \lambda_1 + \cdots + \lambda_m \geqslant C\lambda_1 m^2.$$

As a result, we obtain

$$\mathrm{Tr}(\tilde{P}_m(s) \circ F'(\tilde{T}(s)) \circ \tilde{P}_m(s)) \leqslant -C\lambda_1 m^2 + C[\|\tilde{T}(s)\|^2 + \|T^*\|^2_{H^1(M)} + |\tilde{T}|^6\|\nabla\tilde{T}\|^2].$$

Hence,

$$\frac{1}{t}\int_0^t \mathrm{Tr}(\tilde{P}_m(s)\circ F'(\tilde{T}(s))\circ \tilde{P}_m(s))\,\mathrm{d}s$$
$$\leqslant -C\lambda_1 m^2 + C\left[\frac{1}{t}\int_0^t \|\tilde{T}(s)\|^2\,\mathrm{d}s + \|T^*\|^2_{H^1(M)} + \frac{1}{t}\int_0^t |\tilde{T}|^6\|\nabla\tilde{T}\|^2\,\mathrm{d}s\right].$$

Therefore, by applying (80), we get

$$\limsup_{t\to\infty}\ \sup_{|\tilde{T}_0|\leqslant R_a(T^*,Q)}\ \sup_{\substack{\zeta_j\in L^2(\Omega)\\ |\zeta_j|\leqslant 1\\ j=1,\ldots,m}}\left[\frac{1}{t}\int_0^t \mathrm{Tr}(\tilde{P}_m(s)\circ F'(\tilde{T}(s)))\circ \tilde{P}_m(s)\,\mathrm{d}s\right]$$
$$\leqslant -C\lambda_1 m^2 + K_4(T^*, Q),$$

where

$$K_4(T^*, Q) = C(R_a(Q, T^*))^6[1 + \|T^*\|^2_{H^1(M)} + |Q|^2]. \tag{158}$$

In order to guarantee $-C\lambda_1 m^2 + K_4(T^*, Q) \leqslant 0$ we need to choose $m$ large enough such that

$$m > C\left(\frac{K_4(T^*, Q)}{\lambda_1}\right)^{1/2}.$$

Therefore, the Hausdorff and fractal dimensions of the attractor $\mathcal{A}$ can be estimated by (cf, e.g., [7])

$$d_H(\mathcal{A}) \leqslant d_F(\mathcal{A}) \leqslant C\left(\frac{K_4(T^*, Q)}{\lambda_1}\right)^{1/2}. \qquad \square$$

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