

## Chapter Two

### THE TIME VALUE OF MONEY – Conventions & Definitions

#### Introduction

Now, we are going to learn one of the most important topics in finance, that is, the time value of money. Note that almost every course, which you will take as finance major, depends largely on the time value of money. Hence, it is a good idea to spend a fair amount of time in learning the concepts.

Essentially, we will learn the following concepts:

1. The conventions used in the study of time value of money

2. The time value of money under simple rate of interest

The simple rate of interest nowadays is mostly of academic interest. You will seldom find any transaction either in the real world or in the academics that is based on the simple rate of interest. In fact, this is a fortunate development in the sense that the only thing that is simple about the simple rate of interest is its name. Otherwise the mathematical foundations and the resultant applications are almost impossible to deal with mathematically. Just to understand its complexity we will devote some time on this topic.

In the simple rate of interest we will learn

- a. Future value of an amount
- b. Present value of an amount
- c. Future value of an annuity
- d. Present value of an annuity

3. Compound rate of interest

All the topics in the time value of money that we will learn are under compound rate of interest. The topic that will be covered can be broadly categorized as in two main categories

- a. The time value concepts under the lump sum case
- b. The time value concepts under a series of payments case

In lump sum case we will learn

- a. Present value of an amount
- b. Future value of an amount
- c. Finding the unknown rate of interest
- d. Finding the unknown time period

Under the series of payments topic we will learn present and future value of a series of payments including future and present value of annuities and annuity

dues. We will also learn how to find the unknown rate of interest as well as the unknown time period. All of the above concepts will be dealt with under annual compounding, compounding ‘ $m$ ’ ( $m > 1$ ) times per year and continuous compounding.

#### 4. Special topics

Under this we will discuss the concept of time value of money for such topics but not limited to cash flows growing at a constant rate (under the constant growth rate) and/or under constant level increments, when compounding and deposit intervals are different etc.

### CONVENTIONS:

Conventions mean that unless mentioned otherwise the governing rules follow the conventions.

For example,( READ CAREFULLY)

Whether the problem tells you or not, if it is not mentioned that whether the deposits (withdrawals) are made at the end or the beginning of the period; it (deposits or withdrawals) is always assumed to be done at the end of the period.

For example, consider the following statement

“Deposits of \$200 and \$400 will be made in year 3 and 4”

This statement doesn’t say whether the deposit will be made at the end or at the beginning of the period. Hence, it will always be assumed that the deposits are made at the end of the period. Had the deposit been made at the beginning of the year, the statement would have been “Deposits of \$200 and \$400 will be made at the beginning of years 3 and 4”.

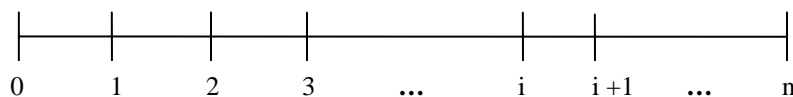
Having said that we enumerate the conventions as follows:

1. All the transactions are done at the end of the period basis unless stated otherwise

#### 2. Time Line

Time line is simply a straight line with numbers  $0, 1, 2, 3 \dots n$  written under it.

A typical time line will look like



Each of the points 1,2,3... stands for the **end** of the period 1,2,3... that is, in the above diagram 3 should be read as end of period 3. The beginning of period 3 will be at point 2, which is the end of period 2.

The point 0 is the present or beginning of period 1. Similarly, the point n is the end of period n. The interval between i and i+1 should be read as “during the period i+1”.

For example, in the above diagram the interval between one and two should be read as during period 2.

3. Note that in number 2 we did not specify any particular “time unit” for the word period. Depending upon the problem at hand the period could be one year, one month, ten minutes, one million years. The general tendency is that when somebody mentions the word “period”, the students usually assume the period as one year long. Never assume that unless it is the rate of interest.
4. The rate of interest is always quoted in annual terms whether it is specifically mentioned or not.

For example, if you call a mortgage broker to ask him/her the current rate of interest on a 30 year mortgage. He/she might say 4% without specifying 4% per year or per month or whatever. The very statement “rate of interest 4%” means that it is the annual rate. If it is not an annual rate, it will say so specified specifically. For example, if you borrow \$100 from Mr. “machine-gun” Hernandez, the local crime boss, he might inform you “hey man, the raaate is 9 peercent per weeeek, comprehende”. Then it is, of course, 9 percent per week.

5. In the computational formula (will come later), the rate of interest always enters in fractions.

For example,

10 % will enter as 0.10

0.5 % will enter as 0.005

If you are confused as what fraction to use, just divide the annual percentage rate by 100.

For example,

0.2% will be equal to 0.2/100 is 0.002 etc.

6. However, unlike the above convention (number 5), in case of using calculator the rate of interest is entered as percent.

For example,

10% rate of interest one will enter 10

These are the standard conventions used almost all the times. There are some specific conventions related to a particular topic. Those will be mentioned at appropriate times.

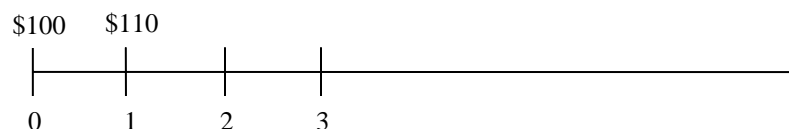
Before we start discussing the actual computational procedure we introduce the following two definitions. For the time being we assume **that period is one year long**.

7. Two aspects of the future value of an annuity computation should be kept in mind. First, by convention, the future value of an annuity is computed just after the last deposit unless stated otherwise; and second, the number of interest earning periods is one less than the number of deposits. In the previous discussion, (n) deposits of an amount P have been made at one-year intervals, but the number of interest-earning periods is only (n-1).

In case of the annuity-due the number of interest earning periods is the same as the number of deposits. That is, if there are n deposits, then the number of interest-earning periods is also n. For example, suppose one is going to make n deposits over an n-year period, and wants to compute the FV of the account at the end of year n. If all deposits are made at the end-of-the-year, then the FVA formula should be used since the FV is computed right after the past deposit is made, the end of year n. However, if all the deposits are made in the beginning of the year, then the FVAD formula should be used because the FV is computed one year after the last deposit. It is widely argued that the FVA formula is used when deposits are end-of-year, and FVAD formula is adopted when deposits are beginning-of-year. While this argument is not invalid, the application of the formula should be based on the timing of the FV computation rather than the timing of the deposits.

### **Definition 1 - The simple rate of interest:**

In the simple of interest only the principal earns interest in every period over the life of the amount. The interest earned at the end of the period on the principal will not earn interest in any of the subsequent periods



Consider the above time line. You deposit \$100 at time 0 and assuming the rate of interest is 10% per annum. At the end of period 1, you will have \$110 in the account. The breakdown of \$110 is as follows:

$$\begin{aligned} &\text{Principal } \$100 \\ &\text{Interest earned} = 0.10 \times \$100 = \$ 10 \\ &\text{Total amount} = \$100 + \$10 = \$110 \end{aligned}$$

Under the simple rate of interest in the successive period only the principal (\$100) will earn the interest. The interest earned i.e. \$10 in period 1 will not earn interest in any of the subsequent periods. Therefore, the total amount in the account at the end of the period 2 will be \$120.

i.e.  $\$100 + \$10 + \$10 = \$120$  computed as follows

$$\begin{aligned} &\text{Principal } \$100 \\ &\text{Interest earned in period 1} = 0.10 \times \$100 = \$ 10 \\ &\text{Interest earned in period 2} = 0.10 \times \$100 = \$ 10 \\ &\text{Total amount} = \$100 + \$10 + \$10 = \$120 \end{aligned}$$

**Definition 2 – Compound rate of interest:**

In compound rate of interest, the interest earned on the principal during the specified period also earns interest in the subsequent periods.

For example:

In the above diagram at the end of period 1, the account will have \$110 under both approaches (simple interest and compound interest). But at the end of period 2, the total amount will be \$121 under the compound interest approach rather than \$120 as we obtained in the simple interest approach.

Break down is as follows:

$$\begin{aligned} &\text{Principal } \$100 \\ &\text{Interest earned in period 1} = 0.10 \times \$100 = \$ 10 \\ &\text{Interest earned in period 2} = 0.10 \times \$100 + 0.10 \times \$10 = \$ 11 \\ &\text{Total amount} = \$100 + \$10 + \$11 = \$121 \end{aligned}$$

Since compound rate of interest, interest earned also starts earning interest the total amount becomes \$121. The difference of \$1 between compound interest and simple interest approach is due to interest of \$1 earned in period 2 on \$10 interest that was earned in period 2.

**Definition 3 – Compounding period:**

The so-called “specified interval” is called the compounding interval.

For example,

“Annual compounding” means that the interest earned during the year also starts earning interest in the next period, that is, at the end of one year. Similarly, monthly compounding means that the interest earned during the one-month period will also start earning interest at the end of the month.

**Definition 4**

Unless stated otherwise, a “calendar time period” will be defined as follows:

- 1 year = 12 months
- 1 year = 365 days
- 1 year = 52 weeks
- 1 year = 4 quarters
- 1 quarter = 3 months
- 1 month = 4 weeks
- 1 month = 30 days

Note that the above calendar definition of time period is a standard. It is specified because there is a discrepancy in these definitions.

For example,

If month = 4 weeks and year = 12 months, then ideally there should be 48 weeks in a year. But, in computations these things are usually not taken into account unless the problem asks you to specifically account for that.

**Definition 5**

In the process “compounding” we move from present to future.

**Definition 6**

In the process “discounting”, we move from future to present.

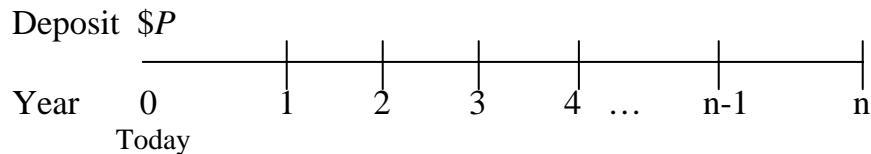
Having equipped ourselves with these conventions and definitions we now proceed to the topic of simple rate of interest.

## Chapter Three

### SIMPLE RATE OF INTEREST

#### FUTURE VALUE OF AN AMOUNT

Let a principal amount  $\$P$  be deposited in an account that has a stated or nominal interest rate of  $k$  or  $100k$  percent per annum. What will be the future value of these  $P$  dollars if left in the account for  $n$  years? The mathematical basis can be developed with the help of the following diagram:



Remembering that with simple interest only the principal ( $P$ , in this case) earns interest, we have the following formulas for interest earned at the end of

$$\begin{aligned} \text{Year 1} &= kP \\ \text{Year 2} &= 2kP \\ \text{Year 3} &= 3kP \\ &\cdot \\ &\cdot \\ \text{Year } n &= nkP \end{aligned}$$

Therefore, the principal  $P$  will grow to the future value ( $FV$ ) as follows:

$$FV = P + nkP = P(1 + nk) \quad (3.1)$$

The future value ( $FV$ ) of an amount  $P$  at an annual simple interest rate of  $k$  for  $n$  years is given by

$$FV = P(1 + nk).$$

What if the interest rate does not stay the same over time? For example, the simple interest rate is  $k_1$ , for the first  $n_1$  years,  $k_2$  for the next  $n_2$  years,  $k_3$  for the next  $n_3$  years, etc. Then, the principal  $P$  will accrue interest  $Pk_1n_1$  for the first  $n_1$  years,  $Pk_2n_2$  for the

next  $n_2$  years, and  $Pk_3n_3$  for the next  $n_3$  years, etc. This implies that the future value under changing simple interest rates is

$$FV = P(1 + n_1k_1 + n_2k_2 + n_3k_3 + \dots)$$

Example 3.1: Suppose an account earns 15 percent simple interest per year. What would the future value be of a deposit of:

- (a) \$15 left for 10 years?
- (b) \$0.15 left for 5 years?
- (c) \$1 left for 2,000 years?

Answers

(a)  $P = 15, k = .15, n = 10$

$$FV = 15(1 + .15 \times 10) = \$37.5$$

(b)  $P = .15, k = .15, n = 5$

$$FV = .15(1 + .15 \times 5) = \$0.2625$$

(c)  $P = 1, k = .15, n = 2,000$

$$FV = 1(1 + .15 \times 2,000) = \$301$$

Example 3.2: Suppose you deposit \$1,000 in an account pays simple interest. What will be the future value of the account if:

- (a) the annual simple interest rate is 7% for the first 5 years, 10% for the next 10 years, and 12% for the last 5 years?
- (b) 5% for the first 10 years, 10% for the next 10 years, 15% for the last 10 years?

Answers:

(a)  $P = \$1,000, k_1 = 7\%, n_1 = 5$   
 $k_2 = 10\%, n_2 = 10$   
 $k_3 = 12\%, n_3 = 5$

$$FV = \$1,000(1 + .07 \times 5 + .10 \times 10 + .12 \times 5)$$

$$= \$1,000(2.95) = \$2,950.$$

(b)  $P = \$1,000, k_1 = 5\%, n_1 = 10$   
 $k_2 = 10\%, n_2 = 10$   
 $k_3 = 15\%, n_3 = 10$

$$\begin{aligned}
 FV &= \$1,000(1 + .05 \times 10 + .10 \times 10 + .15 \times 10) \\
 &= \$1,000(4) = \$4,000.
 \end{aligned}$$

### PRESENT VALUE OF AN AMOUNT

The formula for the present value of some assets to be received in the future can be directly obtained from expression (3.1). For an amount  $FV$  to be received after  $n$  years, the present value is  $P$ -the amount initially deposited in the account. Thus, solving for  $P$  and calling it instead the present value,  $PV$ , we have

$$PV = \frac{F}{1 + nk} \quad (3.2)$$

In the above expression,  $F$  is equal to  $FV$ , the amount to be received after  $n$  years.

The present value ( $PV$ ) of an amount  $F$  received at the end of  $n$  years with a simple rate of interest  $k$  is given by

$$PV = \frac{F}{1 + nk}$$

The formula in the box assumes that rate of interest  $k$  will remain constant in each of the periods. However, if there are more than one simple interest rate over time, that is,  $k_1, n_1, k_2, n_2, k_3, n_3, \dots$  etc. Then

$$PV = \frac{F}{1 + n_1k_1 + n_2k_2 + n_3k_3 + \dots}$$

Example 3.3: Suppose an account carries a 15 percent annual simple interest rate. What will the present value be of:

- (a) \$37.50 to be received at the end of 10 years?
- (b) \$0.2625 to be received at the end of 5 years?
- (c) \$301 to be received at the end of 2,000 years?

Answers:

- (a)  $F = 37.50, k = .15, n = 10$

$$PV = \frac{37.5}{1 + .15 \times 10} = \$15$$

(b)  $F = 0.2625, k = .15, n = 5$

$$PV = \frac{0.2625}{1 + .15 \times 5} = \$0.15$$

(c)  $F = 301, k = .15, n = 2000$

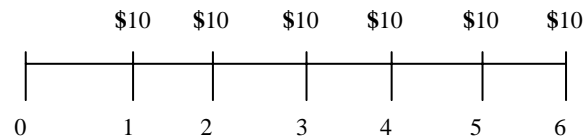
$$PV = \frac{301}{1 + .15 \times 2000} = \$1.00$$

One easily can see that the answers correspond to the original principal balances in the previous example on future value.

### Definition of an Annuity

The problems that we considered above come under the lump sum case. Below we discuss the procedure to obtain the present and future value of series of payments. Here also, we need to delineate a few conventions.

Consider the following time line.



In the above time line deposits of \$10 are made at the end of each year. As per the convention stated above, the above statement “Deposits of \$10 are made each year for six years”.

We define a sequence of deposits as an annuity if it satisfies the following three conditions

1. The deposits are of the same amount, known as level deposits in each period
2. The deposits must be made at equidistant intervals.

For example,

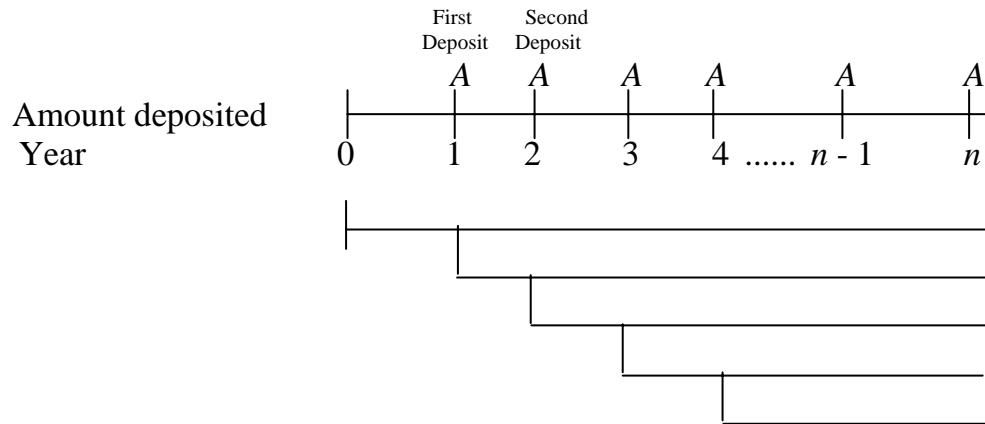
Every year, every month, every ten minutes, etc

3. If it is a compound rate of interest problem then compounding interval must synchronize with deposit intervals.

Note that in the simple rate of interest the question of the above statement (number 3) doesn't arise.

## Future Value of an Annuity

We are, of course, interested in the future value of a series of level deposits at the end of  $n$  years if the account earns an annual simple interest rate of  $k$ . The mathematical formula can be obtained with the help of the following diagram.



Applying formula (3.1) to each of the deposits, we have

$$FV \text{ of first } A = A[1 + (n-1)k]$$

$$FV \text{ of second } A = A[1 + (n-2)k]$$

$$FV \text{ of third } A = A[1 + (n-3)k]$$

$$\vdots$$

$$FV \text{ of } (n-1)^{\text{th}} A = A[1 + k]$$

$$FV \text{ of } n^{\text{th}} A = A$$

Thus the future value of this annuity will be

$$\begin{aligned} FVA &= A[1 + (n-1)k] + A[1 + (n-2)k] + \cdots + A(1 + 2k) + A(1 + k) + A \\ &= A + A + \cdots + A + Ak[(n-1) + (n-2) + \cdots + 2 + 1] \\ &= nA + Ak[(n-1) + (n-2) + \cdots + 2 + 1] \\ &= nA + Ak \left[ \frac{n(n-1)}{2} \right] \\ &= nA \left[ 1 + \frac{(n-1)k}{2} \right] \end{aligned} \tag{3.3}$$

The future value of an annuity (*FVA*) of  $n$  deposits made at the beginning or the end of each year at simple rate of interest  $k$  is given by

$$FVA = nA \left[ 1 + \frac{(n-1)k}{2} \right]$$

Another commonly used terminology, the future value of an annuity due, is computed one period after the last deposit is made. We compute the Future Value of Annuity due as

$$\begin{aligned} FVAD &= A(1+nk) + A[1+(n-1)k] + A[1+(n-2)k] + \dots + A(1+k) \\ &= nA + Ak[n+(n-1)+(n-2)+\dots+1] \\ &= nA + Ak \left[ \frac{n(n+1)}{2} \right] \\ &= nA \left[ 1 + \frac{(n+1)k}{2} \right] \end{aligned} \quad (3.4)$$

**Example 3.4:** Suppose an account earns a 15 percent simple rate of interest annually. What will the future value be of an annual deposit of:

- (a) \$15 at the end of each year for four years?
- (b) \$15 at the beginning of each year for four years?

Since the number of deposits is not too large, we can obtain the answer in two ways-either directly or by using the expression (3.3).

**Answer: Direct method:**

	15	15	15	15
Amount deposited				
End of year	0	1	2	3
	----->	----->	----->	----->

$$\begin{aligned} FVA &= 15(1 + .15 \times 3) + 15(1 + .15 \times 2) + 15(1 + .15) + 15 \\ &= 21.75 + 19.50 + 17.25 + 15 = \$73.50 \end{aligned}$$

Using expression (3.3),

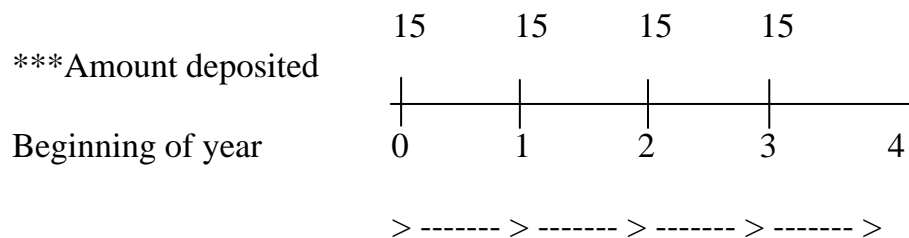
$$\text{Number of deposits} = n = 4$$

$$\text{Number of interest earning periods} = (n - 1) = 3$$

$$A = 15; k = .15$$

$$\begin{aligned} FVA &= nA \left[ 1 + \frac{(n-1)k}{2} \right] \\ &= (4)(15) \left[ 1 + \frac{3 \times .15}{2} \right] \\ &= \$73.50 \end{aligned}$$

For example (b), we compute the value as follows: Direct method:



$$FVA = 15(1 + .15 \times 3) + 15(1 + .15 \times 2) + 15(1 + .15) + 15$$

Using expression (3.3),

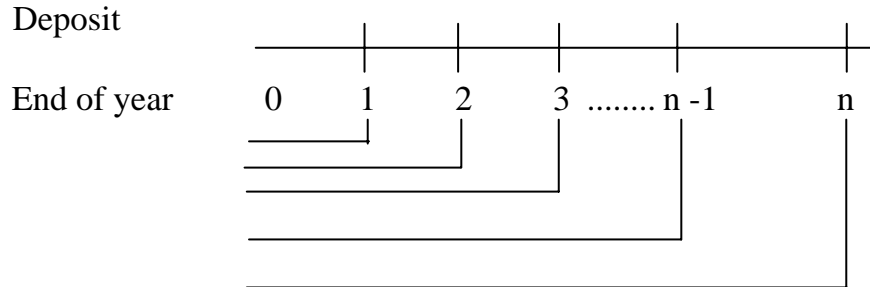
$$FVA = nA \left[ 1 + \frac{(n-1)k}{2} \right] = (4)(15) \left[ 1 + \frac{3 \times .15}{2} \right] = \$73.50.$$

Notice that in computing the future value of an annuity it does not matter whether the deposits are made at the end or the beginning of the period as long as the number of deposits remains the same.

### Present Value of an Annuity

We basically reverse the process just described to obtain the present value of an annuity. Let an amount  $A$  be deposited at the *end* of each year in an account that earns simple interest  $k$  annually. What will be the present value of this annuity? The mathematical formula can be obtained with the help of the following diagram:

A    A    A    A    A



Applying formula (3.2) to each deposit of  $A$  we have

$$PVA^e = \frac{A}{(1+k)} + \frac{A}{(1+2k)} + \frac{A}{(1+3k)} + \cdots + \frac{A}{(1+nk)} \quad (3.5)$$

No further simplification is possible. The superscript  $e$  denotes that the deposits are made at the end of each year. If the deposits are made at the beginning of each year, the formula for a total of  $n$  deposits would be

$$PVA^e = A + \frac{A}{(1+k)} + \frac{A}{(1+2k)} + \cdots + \frac{A}{[1+(n-1)k]} \quad (3.6)$$

Again, no further simplification is possible.

As stated earlier, the only thing simple about the simple rate of interest is its name. Note that most of the real world transactions in life are governed by present value or annuity paradigm.

For example,

The mortgage payments on your house or lease payment on your car is computed on the basis of compound rate of interest

As you see the formula 2.4 and 2.5 are not capable of any further simplification. It will not be possible to deal with them in the case of 30-year loan with monthly repayments. This statement will be clear to you when we deal with compound rate of interest problem in the next section.

The present value of  $n$  annual deposits of  $A$  dollars ( $PVA$ ) with a simple rate of interest  $k$  is given by

$$PVA^e = \frac{A}{(1+k)} + \frac{A}{(1+2k)} + \dots + \frac{A}{(1+nk)}$$

if the deposits are made at the end of each year, and

$$PVA^b = A + \frac{A}{(1+k)} + \dots + \frac{A}{[1+(n-1)k]}$$

if the deposits are made at the beginning of each year.

It should be noted that  $PVA^e$  is simply called the present value of an annuity, whereas  $PVA^b$  is sometimes referred to as the present value of an annuity due.

### Problems with solutions on Simple Rate of Interest

- 1) Future value of \$10 deposit earning simple rate of interest of 7.6% per year at the end of the 5<sup>th</sup> year.

$$FV = 10[1 + (0.076)(5)] = \$13.80$$

- 2) Payment of \$100 at the beginning of year 10. If the simple rate of interest is 8.3%, what is the present value?

$$PV = \frac{100}{[1 + (9)(0.083)]}$$

$$PV = \$57.24$$

- 3) In how many years \$100 will become \$260 if  $k = 0.5\%$  per year?

$$n = \frac{\left(\frac{260}{100}\right) - 1}{.005} = 320 \text{ years}$$

- 4) At what rate of interest an amount will double in 29 years?

$$k = \frac{2-1}{29} = 0.03448 = 3.448\%$$

- 5) Deposits of \$10 are made in an account for 20 years. If  $k = 8\%$  per year. How much will you have in your account at the end of 50 years?

$$FVA = (10)(20) \left[ 1 + \frac{(19)(0.08)}{2} \right]$$

$$FVA = \$352 + [(10)(20)(1 + (0.08)(30))]$$

$$FVA = \$352 + \$680 = \$1,032$$

- 6) The deposit of \$8 is made in an account for 20 years. If the rate of interest is 6%, obtain the present value of this annuity as well as annuity due:

$$PVA = \frac{8}{1+0.06} + \frac{8}{1+2(0.06)} + \frac{8}{1+3(0.06)} + \dots + \frac{8}{1+20(0.06)}$$

$$PVAD = 8 + \frac{8}{1+0.06} + \frac{8}{1+2(0.06)} + \frac{8}{1+3(0.06)} + \dots + \frac{8}{1+19(0.06)}$$

- 7) In how many years will \$100 become \$320 if  $k = 11\%$ ?

$$n = \frac{\left( \frac{320}{100} - 1 \right)}{0.11} = 20 \text{ years}$$

You must have read and understood the discussion and should be able to do the problems as well. If however, if you can do the following problems without looking at the solutions consider yourself well versed in the time value of money under the simple rate of interest.

*Problem 1:*

You make a deposit of \$50 each year for 50 years in an account that earns a simple rate of interest of 5% per year. Assuming that you make the first deposit today what will be the future value of the deposits in 100 years.

*(Hint: Before you embark on the solution to the problem keep in mind that in simple rate of interest only the principal earns interest. Furthermore, you are making 50 deposits in an account beginning today, i.e., at point 0. Therefore, you must be careful in the counting the number of periods.)*

As noted earlier, the topic of simple rate of interest is of academic interest only. Therefore, we will not devote any more time in simple rate of interest.

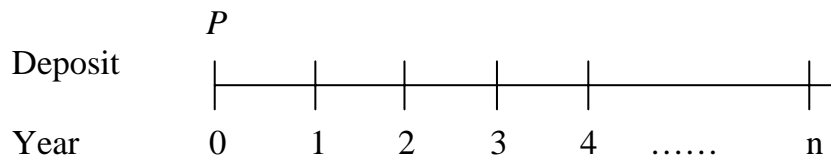
## Chapter Four

### THE TIME VALUE OF MONEY WITH ANNUAL COMPOUNDING

By convention, the interest rate is always quoted in annual terms, but the compounding interval can be of any length of time. Typically, it is less than or equal to a year. The mathematical formulas that follow determine future and present values with compounded rates that assume **ANNUAL COMPOUNDING**.

#### Future Value of an Amount

Let an amount  $P$  be deposited in an account that earns an annual rate of interest  $k$ . What will its future value be at the end of  $n$  years? The mathematical formula can be developed as follows:



Since the interest earned on principal  $P$  together with the principal starts earning interest  $k$  at the end of each year, we have the following situation:

Principal at the end of the year:

0 (today)	$P$	
1	$P + kP$	$= P(1 + k)$
2	$P(1+k) + kP(1+k)$	$= P(1 + k)^2$
3	$P(1+k)^2 + kP(1+k)^2$	$= P(1 + k)^3$
.		
.		
.		
$n$	$P(1+k)^{n-1} + kP(1+k)^{n-1}$	$= P(1 + k)^n$

But the amount at the end of the  $n$ th year is the future value of  $\$P$  deposited today. Therefore, the future value of  $P$  is

$$FV = P(1 + k)^n. \quad (4.1)$$

The future value ( $FV$ ) at the end of  $n$  years of an amount  $P$  deposited today at an annual rate of interest  $k$  compounded annually is

$$FV = P(1 + k)^n.$$

Example 4.1: Suppose at the time of your birth, 25 years ago, your father deposited \$1,200 in an account at an annual interest rate of 15 percent. How much money would exist in that account today?

*Answer:*  $P = 1,200$ ,  $k = .15$ ,  $n = 25$

$$\begin{aligned} FV &= 1,200(1 + .15)^{25} \\ &= \$39,502.74. \end{aligned}$$

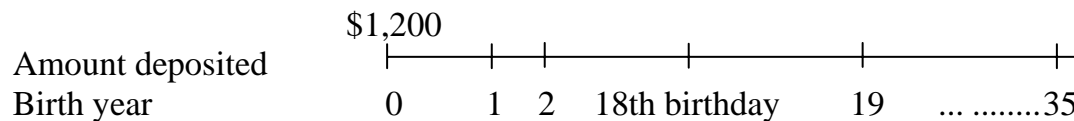
This is a tidy sum to say the least.

Example 4.2: Suppose at the time of your birth, 35 years ago, your father deposited \$1,200 in an account that earns an annual interest rate of 15 percent with the following stipulations:

- (a) You must withdraw half the accumulated amount on your eighteenth birthday;  
and
- (b) On or after your thirty-fifth birthday, you can close the account.

Today is your thirty-fifth birthday. How much money is left in the account to withdraw?

Answer:



The amount of money in the account on the eighteenth birthday is:

$$FV = 1,200(1 + .15)^{18} = \$14,850.54.$$

Since half of that must be withdrawn:

$$\text{Amount withdrawn} = 14,850.54 \div 2 = \$7,425.27.$$

The future value of the remaining balance of \$7,425.27, which by the thirty-seventh birthday will have accrued interest for 17 years, is

$$FV = 7,425.27(1 + .15)^{17} = \$79,905.29.$$

Notice how the amount grows at an exponential rate. The last 17 years accumulate nearly 5 times as much as the first 18 years.

**Example 4.3:** Suppose Mr. Smith contracted with Roles Rice, Inc., to take delivery on a car in four years for \$100,000. An initial deposit of \$34,000 was paid at the time of the contract, and the rest will be paid at the time of delivery (four years from now). Mr. Smith went to his bank and deposited a sum, \$A. He made a promise to himself that he would withdraw half the accumulated amount from this account each year. At the time of delivery, there must be exactly \$66,000 in the account to pay for the car. How much money must Mr. Smith deposit in his account? (Assume a 15 percent annual interest rate.)

**Answer:** The amount remaining at the end of each year withdrawal is shown below:

Year	0	1	2	3	4
Amount	----- ----- ----- ----- -----				
deposited	A	$\frac{A(1+k)}{2}$	$\frac{A(1+k)^2}{2} \times \frac{1}{2}$	$\frac{A(1+k)^3}{4} \times \frac{1}{2}$	$\frac{A(1+k)^4}{8} \times \frac{1}{2} = \$66,000$

Therefore,

$$\frac{A(1+k)^4}{8} = 66,000$$

or

$$A = \frac{66,000 \times 8}{(1+.15)^4} = \$301,885.71$$

Thus Mr. Smith must deposit almost \$302,000 in order to meet his obligation for the car if he goes through with his plan to withdraw half from the account balance each year.

### Present Value of an Amount

In expression (4.1),  $P$  is the present value ( $PV$ ) of a future amount,  $F$ . If we solve equation (4.1) for  $P$ , we obtain the present value of an amount to be received after  $n$  years at an annual interest rate  $k$ . We can write this as

$$PV = \frac{F}{(1+k)^n} \quad (4.2)$$

The present value,  $PV$ , of an amount  $F$  to be received at the end of  $n$  years with an annual rate of interest  $k$  is

$$PV = \frac{F}{(1+k)^n}$$

Some uses of the present value concept follow.

Example 4.4: Joe, Jr., will need \$15,000 to enter college on his eighteenth birthday. How much money must his father deposit on the date of his birth in an account paying a 15 percent rate of interest in order for Joe to have the \$15,000 needed for college?

Answer:  $F = \$15,000$ ;  $n = 18$ ,  $k = .15$ . Therefore,

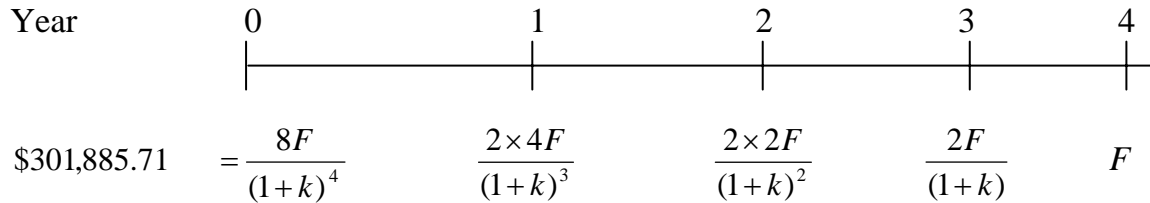
$$PV = \frac{15,000}{(1+.15)^{18}} = \$1,212.08$$

Example 4.5: The Roles Rice, Inc., company made an offer to Mr. Smith, who wanted to buy a customized car. The terms of the offer are as follows:

- (a) The car will be delivered at the end of the fourth year.
- (b) An initial down payment of \$34,000 will be required.
- (c) An initial deposit of \$301,885.71 in a joint account (Roles Rice and Mr. Smith) paying a 15 percent annual rate of interest will also be required.
- (d) Mr. Smith can withdraw half the accumulated amount each year up to the third year. In the fourth year, the company will withdraw the remaining amount.

Mr. Smith accepted the offer. How much money will Roles Rice receive on the delivery date?

Answer: Let the company receive  $\$F$  at the end of the fourth year. Thus the amount of money each year before the withdrawal by Mr. Smith will be



(Read from right to left)

Therefore,

$$F = \frac{\$301,885.71 \times (1 + .15)^4}{8} = \$66,000$$

Again, this problem just reverses the question asked of the future value problem.

### DETERMINING TARGET INTEREST RATES AND PERIODS

So far, we have discussed the ways to obtain the value of assets under various deposit or payment patterns. Sometimes, we are interested in obtaining the number of years needed to reach a given target with an initial deposit or the annual interest rate that must be received for the initial deposit to grow to a given target. The former problem is referred to as the finding of the unknown period, dealt with in this section.

#### Finding the Unknown Period

Suppose a given amount  $P$  is deposited today in an account with an annual rate of interest  $k$ . In how many years will this grow to the target,  $F$ ?

Let  $n$  be the number of years in which  $P$  grows to  $F$ . Then, using expression (4.2), we have

$$P = \frac{F}{(1+k)^n}$$

or

$$F = P(1+k)^n$$

Taking the natural logarithm of both sides yields

$$\ln F = \ln P + n \ln(1+k)$$

or

$$n = \frac{\ln F - \ln P}{\ln(1+k)}$$

or

$$n = \frac{\ln(F/P)}{\ln(1+k)} \quad (4.3)$$

At an annual interest rate  $k$ , the number of years required for an amount  $P$  to grow to a target level,  $T$ , is given by

$$n = \frac{\ln(F/P)}{\ln(1+k)}.$$

### FINDING THE UNKNOWN RATE OF INTEREST

Sometimes one may be interested in knowing at what rate of interest any amount  $P$  deposited today will grow to  $F$  in ' $n$ ' years. In order to solve let us again consider equation 4.2.

$$P = \frac{F}{(1+k)^n}$$

or

$$(1+k)^n = \frac{F}{P}$$

or

$$(1+k) = \left(\frac{F}{P}\right)^{1/n}$$

or

$$k = \left(\frac{F}{P}\right)^{1/n} - 1 \quad (4.4)$$

The unknown rate of  $k$  received for an amount  $P$  to become  $F$  in  $n$  years is

$$k = \left( \frac{F}{P} \right)^{1/n} - 1$$

### Lump Sum Time Value of Money in 5 Minutes

Whatever we have learnt so far under the compound rate of interest can simply be obtained as follows. We developed equation (4.1) as

$$FV = P(1 + k)^n.$$

What did we do?

We knew the value of  $P$ ,  $k$  and  $n$  and obtained  $F$ .

Note that the above is an equation in 4 variables. Therefore, if 3 of them are known one can solve for the unknown as follows.

- a) Known  $P$ ,  $k$ ,  $n$ , what is  $F$ ?

$$F = P(1 + k)^n.$$

- b) Known  $F$ ,  $k$ ,  $n$ , what is  $P$ ?

$$P = \frac{F}{(1 + k)^n}$$

- c) Known  $F$ ,  $P$ ,  $k$ , what is  $n$ ?

$$n = \frac{\ln(F/P)}{\ln(1 + k)}$$

- d) Known  $F$ ,  $P$ ,  $n$ , what is  $k$ ?

$$k = \left(\frac{F}{P}\right)^{1/n} - 1$$

The above derivation tells you how simple it is to deal with time value of money problems if you understand what you are doing.

### Some fun and games with time value of money problems.

An interesting problem that is often found in basic finance textbooks is that of finding how many years it will take for an initial deposit to double at an annual interest rate  $k$ . That is, the problem is to find  $n$  if  $P$  grows to  $2P$  ( $F = 2P$ ) for a given interest rate.

Applying formula (4.3), we have

$$n = \frac{\ln 2}{\ln(1+k)}$$

For  $k < 1$ , we can expand  $\ln(1+k)$  by Taylor's Theorem as shown in the previous chapter.

$$\begin{aligned} \ln(1+k) &= k - \frac{k^2}{2} + \frac{k^3}{3} + \dots \\ &= k \text{ (taking the final approximation)} \end{aligned}$$

Therefore,

$$\begin{aligned} n &= \frac{\ln 2}{k} \\ &= \frac{.6931472}{k} \\ &= \frac{69.31472}{100k} \\ n &= \frac{69}{100k} + a \text{ correction factor} \end{aligned}$$

Since we have approximated  $\ln(1+k)$  with a Taylor series expansion. By trial and error, this correction factor has been found to be approximately 0.35. Thus it will take approximately

$$n = \frac{69}{100k} + .35 \quad (2.12)$$

years for an amount to double itself at any interest rate less than 100 percent. This is known as the rule of 69.

Rule of 69: An amount at annual interest rate  $k$  will double in approximately

$$n = \frac{69}{100k} + .35$$

years where  $100k$  is the rate of interest in *percentage*.

An amount will double in  $n$  years at an approximate annual rate of interest of  $k$ .  
From the above rule of 69 you can also obtain  $k$  if  $n$  is known.

$$k = \frac{100(n - .35)}{69}$$

Note that  $k$  is in fractions.

It should be noted that the value of  $n$ , obtained using the rule of 69, is underestimated for low interest rates, almost the same for intermediate values of the interest rate, and overestimated for higher values of the interest rate. The following example clarifies this point.

Example 4.6: In how many years will \$100 grow to \$200 at annual rates of interest of .005, .01, .05, .10, .15, .20, .25, .30, .40, and .50?

Answer: Using formula (2.12) and the rule of 69, the exact and approximate values of  $n$  for various values of  $k$  are obtained as follows:

$k$	Exact $n$ using equation (2.11)	Approximate $n$ using equation (2.12)	Underestimated (-) or overestimated (+)
.005	138.98	138.35	-0.63
.01	69.66	69.35	-0.31
.05	14.21	14.15	-0.06
.10	7.27	7.25	-0.02
.15	4.95	4.95	0
.20	3.80	3.80	0
.25	3.11	3.11	0
.30	2.64	2.65	+.01
.40	2.06	2.075	+.015
.50	1.71	1.73	+.02

There is another rule of thumb known as the rule of 72 that can be used to approximate the number of years required for an initial amount to double.

Rule of 72: The time required for an initial amount to double at annual interest rate  $k$  is approximately

$$n = \frac{72}{100k}$$

years.

This rule overestimates and underestimates  $n$  according to

$$k \begin{cases} < 1.03874 \ln(I + k) & \text{Overestimate} \\ > 1.03874 \ln(I + k) & \text{Underestimate} \end{cases}$$

Since the overestimation or underestimation is larger than the rule of 69, the rule of 69 is a preferred rule. The following example illustrates this.

Example 4.7: Using the rule of 72, obtain  $n$  for the various values of  $k$  given in the previous example and compare the results.

$k$	Exact $n$ using equation (2.11)	Approximate $n$ using rule of 72	Underestimated or overestimated
.005	138.98	144.00	+5.02
.01	69.66	72.00	+234
.05	14.21	14.40	+ .19
.10	7.27	7.20	- .07
.15	4.95	4.80	- .15
.20	3.80	3.60	- .20
.25	3.11	2.88	- .23
.30	2.64	2.40	- .24
.40	2.06	1.80	- .26
.50	1.71	1.44	- .27

Comparing the last column above with the last column of the previous example, it is obvious that the rule of 69 provides a better approximation than the rule of 72. Another variation of this type of problem is to ask how many years it will take for an amount to

triple at an annual interest rate  $k$ . The exact solution to this problem (or any similar problem) can be obtained using formula (2.11); however, a quick answer can be obtained by using the rule of 110.

*Rule of 110:* An amount at annual interest rate  $k$  will triple in approximately

$$n = \frac{110}{100k} + .52$$

years.

The underestimation or overestimation of  $k \geq .01$  is almost negligible, as can be seen from the following example.

*Example 4.8:* In how many years will \$100 grow to \$300 at annual rates of interest of .005, .01, .02, .05, .10, .15, .20, .25, .30, .35, .40, .50, .75, .90, .95, .99, and 1.00?

*Answer:* The exact and approximate  $n$ 's for various values of  $k$  are given below:

$k$	Exact $n$ using equation (2.11)	Approximate $n$ using rule of 110	Underestimated (-) or overestimated (+)
.005	220.27	220.52	+.25
.01	110.41	110.52	+.11
.02	55.48	55.52	+.04
.05	22.52	22.52	0
.10	11.52	11.52	0
.15	7.86	7.85	-.01
.20	6.03	6.02	-.01
.25	4.93	4.92	-.01
.30	4.18	4.19	-.01
.35	3.66	3.66	0
.40	3.27	3.27	0
.50	2.71	2.72	-.01
.75	1.96	1.98	-.01
.90	1.71	1.74	-.03
.95	1.64	1.66	-.02
.99	1.60	1.63	-.03
1.00	1.58	1.62	-.04