① Separate variables: $\Psi(r,\theta,\phi) = R(r)\,Y(\theta,\phi)$

② $\Longrightarrow$ quantum number n, ℓ, m_ℓ

1) $-ERY$
2) $\cdot r^2$

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + \frac{1}{r^2}\Lambda^2\right)RY + V(r)RY = ERY$$

$$\left[-\frac{\hbar^2}{2\mu}\left(r^2\frac{d^2}{dr^2} + 2r\frac{d}{dr}\right) + r^2(V-E)\right]RY - \frac{\hbar^2}{2\mu}\Lambda^2 RY = 0$$

$\div RY$

$$\underbrace{\frac{1}{R}\left[-\frac{\hbar^2}{2\mu}\left(r^2\frac{d^2}{dr^2} + 2r\frac{d}{dr}\right) + r^2(V-E)\right]R}_{\substack{f(r) \\ = -K}} - \underbrace{\frac{1}{Y}\left(-\frac{\hbar^2}{2\mu}\right)\Lambda^2 Y}_{\substack{g(\theta,\phi) \\ \downarrow \; = +K}} = 0$$

solved before!
$K = \frac{\ell(\ell+1)\hbar^2}{2\mu}$

Y_{ℓ,m_ℓ} are solns

Radial eq:

1) $\div r^2$
2) $+E$
3) $\cdot R$

$$-\frac{\hbar^2}{2\mu}\frac{1}{R}\left(r^2\frac{d^2}{dr^2} + 2r\frac{d}{dr}\right)R + r^2(V-E) + \frac{\ell(\ell+1)\hbar^2}{2\mu} = 0$$

$$-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)R + \underbrace{\left(V + \frac{\ell(\ell+1)\hbar^2}{2\mu r^2}\right)}_{V_{\text{effective}}}R = ER$$

Famous diff. eq., known solns $L_{n,\ell}$

Associated Laguerre polynomials

$$R_{n,l} = N_{n,l}\, \rho^{l} L_{n,l}(\rho)\, e^{-\rho/2}$$

$\rho = \frac{2Z}{n} \cdot \frac{r}{a_0}$ $\qquad$ $a_0 = \frac{4\pi\varepsilon_0 \hbar^2}{m_e e^2}$ $\quad (\mu \approx m_e)$

unitless $\qquad$ "Bohr radius" ≈ 0.529 Å
$= 0.0529$ nm

$L_{n,l}$

$L_{1,0} = 1$

$L_{2,0} = 2 - \rho \qquad L_{2,1} = 1$

$L_{3,0} = 6 - 6\rho + \rho^2 \qquad L_{3,1} = 4 - \rho \qquad L_{3,2} = 1$

$$\psi_{n,l,m_l} = R_{n,l}\, Y_{l,m_l}$$

$n = 1, 2, 3 \ldots$ principal q.n.

$l = 0, 1, 2 \ldots n-1$ orbital angular mom. q.n.

$m_l = 0, \pm 1 \ldots \pm l$ magnetic q.n.

.. "azimuthal"

"bound states" ⇒ discrete

$$E < 0 \qquad E_n = \frac{-Z^2 \mu e^4}{32\pi^2\epsilon_0^2\hbar^2}\cdot\frac{1}{n^2} \doteq -hcZ^2 R_H \cdot \frac{1}{n^2}$$

$$R_H = \frac{1}{hc}\frac{m_e e^4}{32\pi^2\epsilon_0^2\hbar^2} = 1.10\times10^5\ \text{cm}^{-1}$$

"Rydberg constant"

"unbound states" continuum

$E \geq 0$ All values allowed