

RESEARCH DESCRIPTION

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The focus on my research is on the following problems.

- Restriction properties of the Fourier transform and the restriction conjecture,
- Uniform estimates of orthogonal polynomials and special functions,
- Unique continuation properties of solutions of elliptic equations, (and systems),
- Best constants for L^p Fourier multipliers and convolution operators,
- L^p Lebesgue constants and properties of the Dirichlet kernels.

In this note I present these problems, the most significant results that I have obtained so far and the results that I hope to achieve. I will start with a sketchy introduction to these problems and the presentation of my most significant papers. I also have added an Appendix, where the interested readers can find more details over my publications and the research problems that I will discuss

1. The restriction inequality

Let $\mathcal{F}f(\zeta) = \int_{\mathbf{R}^d} f(x)e^{-2\pi i\langle x, \zeta \rangle} dx$ denote the Fourier transform of an integrable function $f(x)$.

Let M be a smooth manifold in \mathbf{R}^d , with $d \geq 2$. If there exist exponents p and q such that

$$(RI) \quad \|\mathcal{F}(f)|_M\|_{L^q(M)} \leq C\|f\|_{L^p(\mathbf{R}^d)}$$

for some constant C independent of f , then we say that M has the (p, q) *restriction property*. The inequality (RI) is the so called (p, q) *restriction inequality*.

1.1 The restriction problem

A natural question to ask is: for which manifolds $M \subset \mathbf{R}^d$ and for which exponents p and q does the inequality (RI) hold ?

The inequality (RI) is always verified when $p = 1$ and $q = \infty$. If M is a manifold of codimension one or higher, Plancherel theorem implies that there exist L^2 functions of norm one whose Fourier transform is arbitrarily large on M , and hence (RI) can never hold when $p = 2$. Moreover, it is not too difficult to see that (RI) can never holds when M lies on a hyperplane and $(p, q) \neq (1, \infty)$.

1.2 The Stein-Tomas restriction theorem

If M does not lie on a hyperplane and $1 < p < 2$, then the restriction properties of M are nontrivial. If $q = 2$ and $M = S^{d-1} = \{x \in \mathbf{R}^d : \|x\| = 1\}$, or more in general, if M is any compact surface in \mathbf{R}^d which has non vanishing Gaussian curvature at every point,¹ then M has the $(p, 2)$ restriction property for every $p \leq p_1 = \frac{2(d+1)}{d+3}$. This is the celebrated *Stein-Tomas restriction theorem*,² (see [To]). Stein-Tomas restriction theorem can be proved in several different manners, but none of these proofs can be generalized to $q < 2$

1.3 Manifolds of codimension 2 or higher, ([DI1], [DI2])

The $(p, 2)$ restriction properties of the Fourier transform to surfaces, and in particular the relationship between the Gaussian curvature and the best exponent p_1 for which the surface has the $(p, 2)$ restriction property, are well understood since many years, but the $(p, 2)$ restriction properties of manifolds of codimension 2 or higher are much less understood.

See the Thesis of M. Christ and G. Mockenhaupt, ([C], [M]), and see also [Pr] and the extensive bibliography of D. Oberlin.

If M is a homogeneous manifold of dimension k ,³ then a well known homogeneity argument due to A. Knapp suggests that the best possible exponent p for which the $(p, 2)$ restriction property holds is

$$p_k = \frac{2(m_1 + \dots + m_s) + 2k}{2(m_1 + \dots + m_s) + k}. \quad (1)$$

This condition is also sufficient in a significant number of cases.

In the joint papers [DI1] and [DI2], (with A. Iosevich), we have proved that the optimal $(p_k, 2)$ restriction inequality (RI) holds for homogeneous manifolds under a variety of conditions on the level sets of the graphing functions $\phi_1, \phi_2, \dots, \phi_s$. The proof of the Theorems in [DI1] and [DI2] rely on accurate non-isotropic decay estimates of the Fourier transform of the measure supported by M . See Appendix 1 for a detailed description of the results proved in these papers.

1.4 The restriction conjecture

The restriction theorems cited so far hold for $q \geq 2$. Indeed, the exponent $q = 2$ plays a crucial role here, since it allows to reduce (RI) to the equivalent “dual” inequality

$$\left\| \int_M \hat{f}(\zeta) e^{2\pi i \langle x, \zeta \rangle} d\sigma(\zeta) \right\|_{L^{p'}(\mathbf{R}^d)} \leq C \|f\|_{L^p(\mathbf{R}^d)},$$

where $d\sigma(\zeta)$ is the Lebesgue measure supported by the manifold M and $\frac{1}{p} + \frac{1}{p'} = 1$. The case $q < 2$ cannot be handled with the same techniques, and it is much more difficult.

¹We recall that the *Gaussian curvature* of a manifold M is the determinant of the Gauss map $J : M \rightarrow S^{d-1}$ taking each point on M to the outward unit normal at that point. The eigenvalues of J are the *principal curvatures* of M .

²Stein’s early result on the restriction theorem are unpublished.

³That is, a manifold of the form of $M = \{(x, x_{k+1}, \dots, x_{k+s}) : x_{k+1} = \phi_1(x), \dots, x_{k+s} = \phi_s(x)\}$, where $x \in \mathbf{R}^k$, $k + s = d$, and, for every $j \leq s$, $\phi_j(x)$ is a smooth and homogeneous function of degree $m_j \geq 1$ on $\mathbf{R}^k \setminus \{0\}$.

The *restriction conjecture* states that S^{d-1} has the (p,q) restriction property if $p < \frac{2d}{d+1}$ and $q \leq \frac{d-1}{d+1}p'$. The conditions on the exponents p and q are necessary, as an example of A. Knapp shows.

The restriction conjecture is perhaps the most important open problem in harmonic analysis. It is deeply connected to important conjectures in Analysis, Partial Differential Equations and Number Theory, the Kakeya conjecture and Bockner-Riesz conjecture being notable examples. See Appendix 2 for a general overview of these conjectures and the progress made toward the solution.

1.5 Partial results on the restriction conjecture, ([DG])

In the joint paper [DG], (with L. Grafakos), we have verified the restriction conjecture in a special class of functions, namely the class \mathcal{C} of the products of radial functions in the Schwartz class $\mathcal{S}(\mathbf{R}^d)$ and “special” spherical harmonics.

The reason why we consider these functions is two-fold. First of all, the linear combinations of functions of \mathcal{C} are dense in $L^p(\mathbf{R}^d)$ for every $1 \leq p \leq 2$. Moreover, \mathcal{C} is invariant under the action of the Fourier transform.

To verify the restriction conjecture in \mathcal{C} we reduced matters to estimating the ratio of the radial parts and the angular parts separately. See Appendix 3 for details.

2 Ultraspherical polynomials and spherical harmonics

The “special” spherical harmonics mentioned in the previous section have an explicit expression in terms of ultraspherical polynomials. The ultraspherical polynomials of degree $j > 0$ and order $s > -\frac{1}{2}$ can be defined as

$$P_j^{(s)}(x) = C_j^\lambda P_j^{(s-\frac{1}{2}, s-\frac{1}{2})}(x),$$

where C_j^s is a normalization constant and

$$P_j^{(\alpha, \beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta} \frac{(-1)^j}{2^j j!} \left(\frac{d^j}{dx^j} \right) (1-x)^{\alpha+j}(1+x)^{\beta+j}$$

is the usual Jacobi polynomial of degree j and order (α, β) on $[-1, 1]$. See [Sz]. The Jacobi polynomials are among the classical orthogonal polynomials which are most used in the applications.

2.1 Uniform estimates for ultraspherical polynomials, ([D1], [D2])

In [D1] I have proved sharp uniform estimates for ultraspherical polynomials of “large” order. That is, I proved that for every $-1 \leq x \leq 1$, $n \geq 0$ and $s \geq n \frac{1+\sqrt{5}}{4}$,

$$\left| \frac{P_n^{(s)}(x)}{P_n^{(s)}(1)} - x^n \right| \leq (1-x^n) \frac{n-1}{2s+1}. \quad (2)$$

The inequality (2) is sharp.

I also estimated $P_n^{(s)}(x)$ in the intervals $[0, z_n^s]$ and $[z_n^s, 1]$, where z_n^s denote the largest zero of $P_n^{(s)}(x)$. In these intervals the uniform estimate (2) can be refined and proved for every $s > 0$ and $n \geq 1$.

These estimates are crucial to prove a number of reverse Hölder inequalities for linear combinations of spherical harmonics and ultraspherical polynomials. See Appendix 4 for details.

2.2 A research project

I am still investigating reverse Hölder inequalities for linear combinations of spherical harmonics and ultraspherical polynomials and their connections with the restriction conjecture. I believe that it is possible to find a reverse Hölder-related condition which implies the restriction inequality (RI), and I am currently investigating the case $d = 2$.

I am also investigating the $L^p - L^q$ behavior of ultraspherical polynomials of “large” degree. These polynomials behave like Bessel functions, in the sense that

$$\lim_{n \rightarrow \infty} \frac{P_n^{(s)}\left(\cos \frac{z}{n}\right)}{P_n^{(s)}(1)} = \Gamma\left(s + \frac{1}{2}\right) \left(\frac{z}{2}\right)^{-s + \frac{1}{2}} J_{s - \frac{1}{2}}(z). \quad (3)$$

(see [Sz], pg. 167). These estimates will complement the ones that I have already obtained in [D1] and [D2], which are sharp when $s \rightarrow \infty$.

3 L^p Lebesgue constants ([AD])

In one dimension there is a very well known estimate

$$\left\| \sum_{|k| \leq N} e(kx) \right\|_1 = \int_{\mathbf{T}^1} \left| \sum_{|k| \leq N} e(kx) \right| dx = \int_0^1 \left| \sum_{|k| \leq N} e(kx) \right| dx \simeq \frac{4}{\pi^2} \ln N, \quad (4)$$

where $e(x) = e^{2\pi i x}$. [Z] This integral is called the Lebesgue constant. In higher dimensions there are as many Lebesgue constants as there are generalizations of an interval of integers, $\{-N, -N + 1, \dots, N\}$.

In the joint paper [AD], (with M. Ash), we have estimated the L^p Lebesgue constants for polyhedra, that is, $\int_{[0,1]^m} |\sum_{k \in NW} e(kx)|^p dx$, when W is a convex polyhedron in \mathbf{R}^m and $p \in (1, \infty)$ and $N \geq 1$.

We have proved that there are constants $\gamma_1(W, p, m)$ and $\gamma_2(W, p, m)$ such that

$$\gamma_1 N^{m(p-1)} \leq \int_{[0,1]^m} \left| \sum_{k \in NW} e(k \cdot x) \right|^p dx \leq \gamma_2 N^{m(p-1)}.$$

Similar results hold for more general regions.

3.1 A research project

A natural question to ask is: how does a L^p Lebesgue constant for a polyhedron depend on the number of sides of the polyhedron in question? This problem is related to the multiplier problem for polyhedra, (see [Co] where the problem has been solved in dimension 2 and for special values of the exponent p). M. Ash and I have partly solved this problem for regular polygons, and we aim to find a solution for the general case.

4 Unique continuation

One of the most important properties of analytic functions is the unique continuation. Indeed, an analytic function which vanishes with all its derivatives at one point of its domain of definition⁴ must vanish identically. This property results from the possibility of expanding an analytic function in power series, and it is usually referred to as *strong unique continuation property*.

Therefore, an analytic function which vanishes in an open subset of its domain of definition must vanish identically. This property is referred to as *unique continuation property*.

These properties are shared by the solutions of a wide class of partial differential equations whose coefficients are not necessarily analytic.

We consider the following general problem: Let Ω be an open and connected set in \mathbf{R}^d , and let $P(x, D)u = 0$ be a system of k linear partial differential equations of order $m \geq 2$ with measurable coefficients. We say that

- i) $P(x, D)$ has the *unique continuation property from an open set*, (or *unique continuation property*), if any two solutions⁵ of the system $P(x, D)u = 0$ which coincide on an open set of Ω coincide everywhere.
- ii) $P(x, D)$ has the *unique continuation property from a point*, (or *strong unique continuation property*), if any two solutions of the system $P(x, D)u = 0$ whose difference vanishes of infinite order at a point $x_0 \in \Omega$ coincide everywhere.

A function u in the Sobolev space $W_{loc}^{2,m}(\Omega, \mathbf{C}^k)$ is said to vanish of infinite order at $x_0 \in \Omega$ if, for every $N > 0$,

$$\lim_{\epsilon \rightarrow 0} \epsilon^{-N} \int_{|x-x_0| < \epsilon} |u(x)|^2 dx = 0. \quad (5)$$

It is easily seen that if $u(x) \in C_0^\infty(\Omega)$, then it vanishes of infinite order at x_0 , in the sense of the previous definition, if and only if all its partial derivatives vanish at x_0 .

Historically, the study of the unique continuation originated from its connection with the question of uniqueness in the Cauchy problem. Indeed, the latter is equivalent to the unique continuation from an open set of the $C^m(\Omega)$ solutions of $P(x, D)u = 0$.

On the other hand, an equally strong motivation came from mathematical physics, the study of the spectral properties of the Schrödinger operator

$$H = -\Delta + V(x), \quad (6)$$

being one of the most important issues.

4.1 Carleman estimates

In 1939 T. Carleman proved in [Car] that the Schrödinger operator (6) has the strong unique continuation property whenever $V(x)$ is locally bounded in \mathbf{R}^2 .

⁴In what follows we will always consider open and connected domains, unless otherwise specified.

⁵A *solution* is a function $u \in W_{loc}^{2,m}(\Omega, \mathbf{C}^k)$ which satisfies $P(x, D)u = 0$ for almost every $x \in \Omega$.

The original proof of Carleman's theorem is based on the establishment of a weighted *a priori* estimate of the form of

$$\|e^{\tau\psi(x)}u\|_{L^2(\mathbf{R}^2)} \leq C\|e^{\tau\psi(x)}\Delta u\|_{L^2(\mathbf{R}^2)}, \quad u \in C_0^\infty(\mathbf{R}^2),^6 \quad (7)$$

where $\psi(x) = -\log|x|$, τ is a positive parameter that is allowed to go to infinity, and C is a constant that does not grow when τ goes to infinity. In Appendix 5 I have shown why an estimate of the form of (7) implies unique continuation properties.

Carleman's original idea has permeated almost all subsequent developments in the subject. Almost all results on unique continuation have been proved using variation of his original idea.⁷

More in general, one can prove the following: Given $P(x, D)u$, a system of linear partial differential operators of order m , with $m \leq d$, which is such that $P(x, D)(0) = 0$, and given exponents $p \geq 1$ and $q \geq 1$ which satisfy the necessary condition

$$\frac{m}{d} = \frac{1}{q} - \frac{1}{p},^8 \quad (8)$$

if there exists an unbounded sequence $\{\tau_j\}_{j \in \mathbf{N}}$ of real positive numbers, and a suitable "weight" function $\psi(x)$ for which the inequality

$$\|e^{\tau_j\psi(x)}u\|_{L^p(\Omega)} \leq C\|e^{\tau_j\psi(x)}P(x, D)u\|_{L^q(\Omega)} \quad (9)$$

holds in $C_0^\infty(\mathbf{R}^d)$, then the operator $P(x, D) - V(x)$ has the unique continuation property, or even the strong unique continuation property, for every vector-valued potential $V(x) \in L_{loc}^r(\Omega, \mathbf{C}^k)$, with $\frac{1}{r} \leq \frac{d}{n}$.

An inequality of the form of (9) is called a *Carleman-type estimate*.⁹ The choice of the weight $\psi(x)$ is crucial for the nature of the result that is implied by (9).

When $V(x)$ is locally bounded and $p = q = 2$, we can prove (9) using more or less sophisticated integration by parts. In the other cases the proof of (9) is much more difficult.

4.2 Second order elliptic operators ([DH])

The tools of the harmonic analysis turned out to be crucial for the establishment of Carleman type inequalities and corresponding unique continuation properties of second order elliptic operators. The literature on the subject is huge, and I will not attempt to survey it. I will only mention the fundamental paper of D. Jerison and C. Kenig, [JK], (with an appendix of E. Stein),¹⁰ and the survey paper of T. Wolff [W] for a basic list of references on the subject.

⁶The original Carleman estimate is actually not quite like the inequality stated here, but the idea is still the same.

⁷See [GL] and [FGL], in which the authors proved unique continuation theorems in $H^{2,1}(\Omega)$ without using Carleman type estimates.

⁸We can prove that condition (8) is necessary using a standard homogeneity argument.

⁹It is worthwhile to remark that if $\Omega = \mathbf{R}^d$, $\psi(x) \equiv 0$, $P(x, D) = \Delta$ and $\frac{1}{q} - \frac{1}{p} = \frac{2}{n}$, (9) reduces to the classical Hardy-Littlewood-Sobolev inequality.

¹⁰D. Jerison and C. Kenig proved that the differential inequality $|\Delta u(x)| \leq V(x)|u(x)|$ has the unique continuation from a point in $W^{2, \frac{2n}{n+2}}(\Omega)$ whenever $V(x)$ is in $L_{loc}^{\frac{n}{2}}(\Omega)$. They also show with a counterexample that $L_{loc}^{\frac{n}{2}}(\Omega)$ cannot be replaced by $L_{loc}^{\frac{n}{2}-\epsilon}(\mathbf{R}^n)$ for any $\epsilon > 0$.

Most of the known results are in $H_{loc}^{2,p}(\Omega)$, with $p \geq \frac{2n}{n+2}$.¹¹

In the joint preprint [DH], (with S. Hudson), we investigated the unique continuation properties of the $H_{loc}^{2,1}(\Omega)$ solutions of the differential inequality,

$$|\Delta u(x)| \leq V(x)|u(x)|, \quad x \in \Omega. \quad (10)$$

and we proved the following theorems.

Theorem 1 *if $d \geq 2$ and $V(x)|x|^2 \leq C$, with $0 < C < \nu(\nu - d + 2)$, then every nonnegative $W^{2,1}(\Omega)$ solution of the differential inequality (10) which vanishes of order $\nu > \max\{d - 2, 1\}$ at some point of Ω is $\equiv 0$.*

Remarkably, the conditions on C cannot be improved, in the sense that, if $C = \nu(\nu - d + 2)$, it is possible to find nontrivial functions that vanish of order ν at the origin which satisfy (10).

4.3 A research project

S. Hudson and I are currently extending our techniques to prove unique continuation properties of more general differential inequalities and equations. We considered in particular the differential inequality $|\operatorname{div}(\lambda(x)\nabla u)(x)| \leq |V(x)u(x)|$, where $\lambda(x)$ is a Lipschitz continuous matrix with nonnegative eigenvalues, and the differential equation $\operatorname{div}(|\nabla u|^{p-1}\nabla u)(x) = 0$. The operator $\operatorname{div}(|\nabla u|^{p-1}\nabla u)$ is the *p-Laplacian*, and is very important in the applications. We are also investigating the properties of the zero set of harmonic functions in subsets of \mathbf{R}^2 .

4.4 A research project

I am investigating the unique continuation properties of the differential inequality equation

$$|\Delta u(x)| \leq V(x)|u(x)|^s + W(x)|\nabla u(x)|^t$$

where $s > 1$ and $t > 1$, and $V(x)$ and $W(x)$ are bounded. The problem is be difficult, and there is almost no literature available. The Carleman estimate technology do no seem to be applicable to nonlinear problems, but I hope that the methods that I am developing with S. Hudson will be powerful enough to handle this problem.

4.5 Elliptic operators of order $m > 2$

While the literature on the unique continuation for elliptic operators of order 2 is extensive, there are not too many papers on the unique continuation of linear partial differential operators of order $m > 2$.

In 1958 P. A. Calderon proved a unique continuation theorem for elliptic operators of order $m > 2$ whose complex characteristics are simple in every direction. See [Cal].

¹¹The exponent $p_0 = \frac{2n}{n+2}$ is remarkable because the Sobolev embedding theorem maps $H_{loc}^{2,p_0}(\Omega)$ into $L^{p'_0}(\Omega)$, where $p'_0 = \frac{2n}{n-2}$ is the dual exponent of p_0 . This property of the exponents is crucial to prove Carleman-type inequalities.

The *complex characteristics* of the operator $P(x, D)$ in the direction $\nu \in S^{d-1}$ are the complex roots of the polynomial $\tau \rightarrow P_m(x, \zeta + i\nu\tau)$, where $P_m(x, \zeta)$ denotes the principal symbol of $P(x, D)$.

The assumption on the characteristics may seem artificial, but is indeed very important. In fact, A. Plis showed in [Pl] that elliptic operators with analytic coefficients and complex characteristics of order six may have nontrivial solutions with compact support.

However, there are cases in which operators with multiple characteristics have the unique continuation property, or even the strong unique continuation property. See [Ok], [CG] and [Gra], just to cite a few.

4.6 Connection with the restriction properties of the Fourier transform, ([D3], [D4], [Dt])

In [D3], I have established the following unique continuation result for a class of operators of the form of $P(D) + V(x)$, where $P(D)$ is an elliptic partial differential operator of order $m \leq n$ with constant coefficients, and $V \in L^{\frac{n}{m}}(\mathbf{R}^d)$.

Theorem 2 *Suppose that $P(D)$ has simple complex characteristics in the direction ν . Then, every $u \in H^{m,p}(\mathbf{R}^d)$, with*

$$\frac{1}{2} \leq \frac{1}{p} \leq \frac{m}{d} + \frac{d-2}{2(d+2)}, \quad (11)$$

which is supported on one side of the hyperplane $\{x : \langle x, \nu \rangle = 0\}$, and solves the differential inequality

$$|P(D)u(x)| \leq |V(x)u(x)|,$$

is $\equiv 0$ in \mathbf{R}^d .

I proved this Theorem using Carleman type estimates. To prove these estimates I used the Christ-Mockenhaupt restriction theorem for manifolds of codimension 2 and complex interpolation, while in [D4] and [Dt] I used the restriction properties of manifolds of codimension 1, which yields Theorem 2 under additional assumptions on the characteristics of $P(D)$. See Appendix 6 for details.

The connection between the restriction properties of hypersurfaces and Carleman inequalities seems to be first observed by L. Hörmander in [H] and has been widely used since then. However, it seems that the restriction properties of manifolds of codimension 2 have been used to prove Carleman type estimates for the first time in my Thesis, [Dt].

4.7 Unique continuation for systems of PDE

Let L be the (n -dimensional) Dirac operator, (see Appendix 7 for the definition and properties of the Dirac operator).

In the joint paper [DO2] (with T. Okaji), we dealt with 2 different problems. The first one concerns the strong unique continuation property of the differential inequality

$$|Lu(x)| \leq |V(x)u(x)|, \quad x \in \Omega \quad (12)$$

and the other one concerns the strong unique continuation property of the differential equation

$$Lu(x) + V(x)u(x) = 0, \quad x \in \Omega. \quad (13)$$

In both cases we consider matrix potentials $V(x)$, (with a possible singularity at the origin), that satisfy

$$|x|V(x) \in L_{loc}^{\infty}(\Omega)^{N^2}, \quad (14)$$

but in the case of the differential equation (17) and when the space dimension is equal to two or three it is possible to weaken the assumption on $V(x)$. See the Appendix 7 for details.

Our results improve theorems of D. Jerison and V. Vogelsang. See [J], [V] and also [KY].

4. Best constants

Let T_m be a Fourier multiplier, that is, a linear operator which is defined on all Schwartz functions f by $T_m f(x) = \int_{\mathbf{R}} m(\xi) \widehat{f}(\xi) e^{2\pi i x \xi} d\xi$, where $m \in L^{\infty}(\mathbf{R})$ and $\widehat{f}(\xi)$ is the Fourier transform of $f(x)$. This can be extended, for some suitable choices of m , to a bounded linear operator which maps $L^p(\mathbf{R})$ into itself. The (p, p) norm of the extended operator is the smallest positive constant c such that $\|T_m f\|_p \leq c \|f\|_p$. We will denote this norm by $\|T_m\|_{p,p}$ or alternatively by $\|m\|_p$.

An important example of a multiplier whose norm is exactly known for all p 's is given by the Hilbert transform H , which corresponds to $m(\xi) = -i \operatorname{sgn}(\xi)$. In [Pi] it is shown that

$$n_p = \|H\|_{p,p} = \|\operatorname{sgn}(\cdot)\|_p = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & \text{if } 1 < p \leq 2, \\ \cot\left(\frac{\pi}{2p}\right) & \text{if } 2 \leq p < \infty. \end{cases} \quad (15)$$

5.1 Description of the papers [DL1] and [DL2]

In the joint paper [DL1], (with E. Laeng), we have shown that the norm n_p in (15) also coincides with the (p, p) norm of the segment multiplier

$$S_{[a,b]} f(x) = \int_a^b \widehat{f}(\xi) e^{2\pi i x \xi} d\xi,$$

where $[a, b]$ is any bounded interval of positive measure. That is in a stark contrast with the results proved by R. Kerman and J. Lang in a more general setting of rearrangement-invariant Banach spaces. Indeed, there exist rearrangement invariant spaces where the Hilbert transform is an unbounded operator but the norm of the segment multiplier is finite.¹² See [KL].

In the joint paper [DL2], (with E. Laeng), we prove the following general theorem concerning the (p, p) norm of Fourier multipliers.

Let $m(\xi)$ be an increasing function on \mathbf{R} and let $\lambda = m(-\infty) = \inf m$, $\Lambda = m(+\infty) = \sup m$. We show that the norms of the corresponding Fourier multiplier, for $1 < p < \infty$, satisfy

¹²In these spaces the Hilbert transform and the segment multiplier are defined as convolution operators.

$|||m|||_p = |||\mu_{\lambda,\Lambda}|||_p$, where $\mu_{\lambda,\Lambda}(\xi)$ is the “two half lines” multiplier defined by

$$\mu_{\lambda,\Lambda}(\xi) = \begin{cases} \Lambda & \text{if } \xi > 0, \\ \lambda & \text{if } \xi < 0. \end{cases}$$

In particular, if $\Lambda = -\lambda$, it follows that $|||m|||_p = |\Lambda|n_p$.

Steckin’s theorem states that if m is a function of bounded variation, then T_m is bounded on $L^p(\mathbf{R})$ for $1 < p < \infty$. See [EG]. Our previous result leads to a proof of Steckin’s theorem with an explicit, (although non sharp), upper bound for $|||m|||_p$.

6 Appendix

A1. Description of the papers [DI1] and [DI2].

The proofs of most restriction theorems in the literature, including Stein-Tomas restriction theorem, relies on accurate estimates of the Fourier transform of the Lebesgue measure supported by the manifold M . Indeed, if M is a compact k -dimensional manifold in \mathbf{R}^d , with $k < d$, and $d\sigma$ is the measure supported by M ,¹³ by a theorem of A. Greenleaf, (see [Gre]), if there exist positive constants C and γ for which

$$|\mathcal{F}[d\sigma](\zeta)| \leq C(1 + |\zeta|)^{-\gamma}, \quad \zeta \in \mathbf{R}^n, \quad (1)$$

then M has the $(p, 2)$ restriction property¹⁴ whenever $p \leq \frac{2(d-k)+2\gamma}{2(d-k)+\gamma}$.

If M is a surface with non vanishing Gaussian curvature at every point, (for example, $M = S^{d-1}$), then (1) holds with $\gamma = \frac{d-1}{2}$. See e.g. [So], and see also the excellent paper [IL], where the Authors shows, among the other things, that the $(p, 2)$ restriction inequality (RI) for smooth compact hypersurfaces S in R^d implies decay properties of the Fourier transform of the measure supported by S .

Therefore, when $q = 2$, the theorem of Greenleaf and the isotropic estimates (1) suffice to prove the optimal restriction properties of surfaces with non vanishing Gaussian curvature at every point.¹⁵

In the paper [DI1] and [DI2] we consider manifolds which are defined as follows.

$$M = \{(x, x_{k+1}, \dots, x_{k+s}) : x_{k+1} = \phi_1(x), \dots, x_{k+s} = \phi_s(x)\}, \quad (2)$$

where $x \in \mathbf{R}^k$, $k + s = d$, and, for every $j \leq s$, $\phi_j(x)$ is a smooth and homogeneous function of degree $m_j \geq 1$ on $\mathbf{R}^k \setminus \{0\}$. Let $m = \max_j \{m_j\}$.

It is not too difficult to prove that the best possible isotropic rate of decay of the Fourier transform of the measure supported by M is $\frac{k}{m}$; therefore, Greenleaf's theorem implies that M has the $(p, 2)$ restriction property for $p \leq \frac{2sm+2k}{2sm+k}$.

However, a well known homogeneity argument due to A. Knapp¹⁶ suggests that these manifolds have the $(p, 2)$ restriction property if

$$p \leq p_s = \frac{2(m_1 + \dots + m_s) + 2d}{2(m_1 + \dots + m_s) + k}, \quad (3)$$

and of course $p_s \geq \frac{2sm+2k}{2sm+k}$. That is an important case in which the isotropic estimates of the Fourier transform of the measure supported by M cannot provide optimal restriction theorems.

For manifolds of codimension 2 or higher one needs to use precise non isotropic decay estimates of the Fourier transform of the measure supported by M , (see [C] and [Pr]).

In the joint papers [DI1] and [DI2], (with A. Iosevich), we have proved that the manifolds defined as in (2) have the optimal $(p_s, 2)$ restriction property, where p_s is as in (3), under a

¹³Sometimes, for technical reasons, $d\sigma$ shall be replaced by $\psi(x)d\sigma$, where ψ is a smooth cutoff function.

¹⁴We recall that M has the (p,q) restriction property if, for every $f \in C_0^\infty(\mathbf{R}^n)$, $\|\mathcal{F}(f)|_M\|_{L^q(M)} \leq C\|f\|_{L^p(\mathbf{R}^d)}$.

¹⁵Indeed, the proof of the theorem of Greenleaf is a modification of E. Stein's original proof of the restriction theorem.

¹⁶A detailed explanation of the argument is in [DI2].

variety of conditions on the level sets of the graphing functions $\phi_1, \phi_2, \dots, \phi_s$. I will cite here a few sample results.

The following Theorem answers a question posed by Fulvio Ricci concerning the restriction theorems for manifolds given as graphs of quadratic monomial. ¹⁷

Theorem 1 *Let $S = \{(x, x_{k+1}, \dots, x_{k+s}) \in \mathbf{R}^k : x_{k+1} = \phi_1(x), \dots, x_{k+s} = \phi_s(x)\}$, where $s = \frac{k(k+1)}{2}$, and the ϕ_j denote the distinct monomial of degree 2. Then the $(p_s, 2)$ restriction property holds.*

The following result generalizes Theorem 1 to manifolds given as joint graphs of smooth functions of order of homogeneity greater than 2.

Theorem 2 *Let M be defined as in (2). Suppose that all the ϕ_j 's are homogeneous of degree $m \geq 2d$ and that they do not vanish simultaneously at any point of S^{d-1} . Suppose also that no linear combination of the ϕ_j 's vanishes on a subset of positive measure of S^{d-1} . Then the $(p_s, 2)$ restriction property holds.*

Note that the exponent p_s in Theorem 1 is $\frac{2(k+2)}{2k+3}$, while the exponent p_s in Theorem 2 is $\frac{2(k+sm)}{k+2sm}$.

The proof of Theorems 1 and 2 and of the other restriction Theorems in [DI1] and [DI2] rely on accurate non-isotropic decay estimates for the Fourier transform of the measure supported by M . A sample result is the following.

Theorem 3 *Let M be as in Theorem 2. Let $\Phi(x) = (\phi_1(x), \dots, \phi_s(x))$, and $\Phi_\lambda(x) = \langle \Phi(x), \lambda \rangle$, where $\lambda \in \mathbf{R}^{s-1}$. Then, for every $\xi \in \mathbf{R}^d$,*

$$\mathcal{F}[d\sigma](\xi, \lambda) \leq C \int_{S^{d-1}} |\Phi_\lambda(\omega)|^{-\frac{k}{m}} d\omega,$$

where C is a positive constant that does not depend on ξ and λ .

A2. The Kakeya conjecture, the Bockner-Riesz conjecture and the restriction conjecture.

The *Kakeya conjecture* states that every Besicovich set of \mathbf{R}^d has Hausdorff and Minkowski dimension d . A Besicovich set is a set which contains a unit segment in every direction.

The Bockner-Riesz conjecture states that, for every $R > 0$, the operators

$$S_\delta^R f(x) = \int_{|\zeta| \leq R} \mathcal{F}f(\zeta) e^{2\pi i x \cdot \zeta} (1 - |\zeta|^2 R^{-2})^\delta, \quad \delta > 0, \quad f \in C_0^\infty(\mathbf{R}^d),$$

extend to bounded operators in $L^p(\mathbf{R}^d)$ whenever $\delta > 0$, and p and δ satisfy the following inequality.

$$d \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{2} < \delta. \quad (4)$$

¹⁷See also Theorem 1 announced in [M1].

Hertz discovered that the condition (4) is necessary, (see e.g. [Graf]). If $p \neq 2$ then the condition $\delta > 0$ is necessary too. In the late 70s C. Fefferman proved in [Fe] that S_0^R is not bounded in any $L^p(\mathbf{R}^d)$ unless $p = 2$. That is equivalent to say that the characteristic function of the ball is not an L^p Fourier multiplier unless $p = 2$.

Fefferman's example is based on a variant of the construction of a Besicovitch set of measure zero, and shows that the Besicovitch sets are intimately connected to the restriction properties of balls and sphere.

When $d = 2$ all these conjectures have been proved. The two dimensional restriction conjecture has been proved by C. Fefferman and E. Stein in 1970, ([Fe1]). Indeed, when $d = 2$, the necessary conditions on p and q stated in Section 1.4 become $p < \frac{4}{3}$ and $q \geq 4$, and the proof in [Fe1] relies on the special properties of the exponent 4. D. Oberlin proved in [Ob] an optimal restriction theorem for plane curves, but also his proof uses the special properties of the exponent 4.

The two dimensional Bockner-Riesz conjecture has been proved, by L. Carleson and P. Sjolin in 1972. See [CS], but also [St] or [Gra]. Carleson and Sjolin's result can be stated as a corollary of a more general theorem concerning oscillatory integrals.

The two dimensional Takeya conjecture has been proved by Davies in 1971, ([Da]).

The Bockner-Riesz conjecture is historically related to the classical problem of convergence of Fourier series and Fourier integrals in dimension $d > 1$.

The Takeya conjecture has been formulated in several different equivalent manners which emphasize the connection with other open problems in Analysis.

Notably, the Takeya conjecture has many interesting combinatorial aspects. See e.g the survey paper of T. Wolff [W1] and the references cited there.

T. Tao proved in [T2] that the Bockner-Riesz conjecture implies the restriction conjecture. It is known that the restriction conjecture implies the Takeya conjecture. The proof is actually not too difficult and uses an equivalent formulation of the Takeya conjecture due to Bourgain, ([B]). It is not clear who first observed the connection between the Takeya conjecture and the restriction conjecture, (see [W1]).

In 1991 J. Bourgain proved that the known result on the Hausdorff and Minkowski dimension of the Besicovitch sets can be used to prove partial results on the restriction conjecture.

Bourgain's pioneering work [B] originated a lot of interest in the Takeya conjecture and its connection with the restriction properties of the Fourier transform.

A lot of progress have been made in recent years, both toward the proof of the Takeya conjecture and the proof of the restriction conjecture, and I will not attempt to survey it. The applicant will only cite the survey papers of T. Tao [T] and T. Wolff [W1], which have a long list of references.

To the best of my knowledge, the partial solution of the restriction conjecture given by T. Tao in [T3] is the best up to this date. He proved that, for $d > 3$, S^{d-1} has the (p, q) restriction property whenever $q \leq \frac{d-1}{d+1}p'$ and $p < \frac{2(d+2)}{d+4}$, and that S^2 has the (p, q) restriction property whenever $q \leq \frac{1}{2}p'$ and $p < \frac{10}{3}$. Note that when $d = 3$ the conjectured upper bound for p is $\frac{3}{2}$.

The proofs of the recent results on the restriction conjecture differ greatly from the proofs of the Stein-Tomas restriction theorem and the Fefferman and Stein's proof of the two dimensional restriction conjecture.

These proofs use rather sophisticated technique involving fine decomposition of the Fourier transform of $f d\sigma$ into wave packets supported on tubes, and local restriction estimates. The known estimates on the dimension of the Besicovitch sets are crucial to obtain control of the

Fourier transform of $f d\sigma$ over certain tubes.

However, even if the Kakeya conjecture was completely solved, the technique which are currently used to deduce restriction properties of S^{d-1} would not yield a complete proof of the restriction conjecture.

Even if the restriction conjecture was proved using Kakeya-related techniques, an alternative proof would still be valuable.

A3. Description of the paper [DG]

The spherical harmonics have an enormous importance in Mathematics and Physics¹⁸.

Let \mathcal{H}_m the space of spherical harmonics of degree m in R^d , $d \geq 2$. A notable set of generators of \mathcal{H}_m is the following.

Consider the usual spherical coordinates on \mathbf{R}^d , $d \geq 2$. In these coordinates, $x \in \mathbf{R}^d$ is the product of $r = |x|$ and $\omega \in S^{d-1}$, and ω can be expressed as a function of $\theta = (\theta_1, \dots, \theta_{d-1})$, with $0 \leq \theta_1, \dots, \theta_{d-2} \leq \pi$, while $0 \leq \theta_{d-1} \leq 2\pi$.

Let $m = m_0 \geq m_1 \geq \dots \geq m_{d-2} \geq 0$ be integers. Let

$$Y_{(m_k)}(\theta_1, \dots, \theta_{d-1}) = e^{\pm i m_{d-1} \theta_{d-1}} \prod_{k=1}^{d-2} (\sin \theta_k)^{m_k} P_{m_{k-1}-m_k}^{(m_k + \frac{d-k-1}{2})}(\cos \theta_k), \quad (5)$$

where $P_n^{(s)}(t)$ is the usual ultraspherical polynomial of degree $n \geq 0$ and order $s > -\frac{1}{2}$.¹⁹

$Y_{(m_k)}$ is a spherical harmonic, and every spherical harmonic in \mathbf{R}^d is a finite linear combination of the $Y_{(m_k)}$'s, (see e.g. [E]). This may be proved using a dimension comparison with the space of the spherical harmonics of degree m which has dimension $(2m + d - 2) \frac{\Gamma(m+d-2)}{\Gamma(m+1)\Gamma(d-1)}$.

In the joint paper [DG] L. Grafakos and I observed that it is enough to prove the restriction conjecture for functions which are linear combination of spherical harmonics in \mathbf{R}^d and smooth radial functions with compact support in $(0, \infty)$. We let

$$\mathcal{C} = \{r^{m_j} f_j(r) Y_j(\omega) : f_j(r) \in C_0^\infty(0, \infty), Y_j(\omega) \in \mathcal{L} \text{ has degree } m_j\}$$

and we observe that $\text{Span}(\mathcal{C})$, the set of finite linear combinations of functions of \mathcal{C} , is dense in $L^p(\mathbf{R}^d)$ for $1 \leq p \leq 2$.

Therefore, the restriction conjecture follows if we prove the restriction inequality (RI) for linear combinations of functions in \mathcal{C} .

In [DG] we proved that (RI) holds in \mathcal{C} for every $1 \leq p < \frac{2d}{d+1} < q \leq \frac{d-1}{d+1} p'$. For these functions, (RI) reduces to

$$\frac{\|\mathcal{F}(r^m f Y_m)|_{r=1}\|_{L^q(\mathbf{S}^{d-1})}}{\|r^m f Y_m\|_{L^p(\mathbf{R}^d)}} \leq C, \quad f \in C_0^\infty(0, +\infty), Y_m \in \mathcal{L}_m, \quad (6)$$

and we can reduced matters to estimating the ratios of the radial parts and the angular parts separately. Indeed, the Fourier transform of $Y_m(\omega) r^m f(r)$ equals

$$C_d i^{-m} Y_m(\xi) \rho^{m - \frac{1}{2}(d+2m-2)} \int_0^\infty f(r) J_{\frac{1}{2}(d+2m-2)}(r\rho) r^{\frac{1}{2}(d+2m)} dr$$

¹⁸Recall that a spherical harmonics in \mathbf{R}^d is the restriction of an homogeneous harmonic polynomial to S^{d-1} .

¹⁹We recall that $P_n^{(s)}(t)$ can be defined as $P_n^{(s)}(t) = C_n^s (1-t^2)^{-s+\frac{1}{2}} \frac{d^n}{dt^n} (1-t^2)^{s-\frac{1}{2}+n}$, where C_n^s is a normalization constant.

where C_d is a constant that depends only on d , (see e.g. [St]). Thus,

$$\mathcal{F}(r^m f Y_m)|_{\rho=1}(\xi) = C_d i^{-m} Y_m(\xi) \int_0^\infty f(r) J_{\frac{1}{2}(d+2m-2)}(r) r^{\frac{1}{2}(d+2m)} dr.$$

Also, $\|r^m f Y_m\|_{L^p(\mathbf{R}^d)} = \|r^{m+\frac{d-1}{p}} f\|_{L^p(0,+\infty)} \|Y_m\|_{L^p(\mathbf{S}^{d-1})}$. For every $p < \frac{2d}{d+1}$ we proved that

$$\begin{aligned} \frac{\left| \int_0^\infty f(r) J_{\frac{1}{2}(d+2m-2)}(r) r^{\frac{1}{2}(d+2m)} dr \right|}{\|r^{m+\frac{d-1}{p}} f\|_{L^p(0,+\infty)}} &\leq \|r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{1}{2}(d+2m-2)}\|_{L^{p'}(0,\infty)} \\ &\leq C m^{(d-1)(\frac{1}{2}-\frac{1}{p})+\frac{1}{p'}}, \end{aligned} \quad (7)$$

and for every $p < q < \frac{d-1}{d+1}p'$, we proved that $\frac{\|Y_m\|_{L^q(\mathbf{S}^{d-1})}}{\|Y_m\|_{L^p(\mathbf{S}^{d-1})}} \leq C m^{(d-2)(\frac{1}{2p}-\frac{1}{2q})}$. Therefore, for every $1 < p < \frac{2d}{d+1} < q \leq \frac{d-1}{d+1}p'$,

$$\sup_{Y \in \mathcal{L}_m} \frac{\|Y_m\|_{L^q(\mathbf{S}^{d-1})}}{\|Y_m\|_{L^p(\mathbf{S}^{d-1})}} \|r^{\frac{d}{2}-\frac{d-1}{p}} J_{\frac{1}{2}(d+2m-2)}(r)\|_{L^{p'}(0,\infty)} < C < \infty, \quad (8)$$

which yield (6).

A4. Reverse Hölder inequalities for spherical harmonics, ([D2])

Reverse Hölder inequality for spherical harmonics are a very important tool in Fourier Analysis. They can be used to prove unique continuation theorem for the differential inequality $|\Delta u(x)| \leq V(x)|u(x)| + W(x)|\nabla u(x)|$ under suitable assumptions on the potentials $V(x)$ and $W(x)$, ([J], [R1], [R2] ...), to prove dimension-free estimates for the Riesz transform, ([DR]), and in many other problems.

Let \mathcal{H}_m the space of spherical harmonics of degree m in R^d , $d \geq 2$. Consider the inequality

$$\frac{\|Y\|_{L^q(S^{d-1})}}{\|Y\|_{L^p(S^{d-1})}} \leq c(m, p, q, d), \quad Y \in \mathcal{H}_m, \quad (9)$$

where $1 \leq p \leq q \leq \infty$ and $c(m, p, q, d)$ does not depend on Y .

C. D. Sogge found a sharp upper bound for the ratio in (9) when $q = 2$ and $d \geq 3$. He proved in [So] that

$$c(m, p, 2, d) = \begin{cases} C(p) m^{(d-1)(\frac{1}{p}-\frac{d}{2(d-1)})} & \text{if } 1 \leq p < \frac{2d}{d+2}, \\ C(p) m^{\frac{d-2}{2}(\frac{1}{p}-\frac{1}{2})} & \text{if } \frac{2d}{d+2} \leq p \leq 2. \end{cases}$$

These estimates are sharp, in the sense that the powers of m cannot be replaced by smaller ones.

In [D2] I proved a reverse Hölder inequality for linear combinations of spherical harmonics. My estimates are sharp, and actually better than the ones in Sogge's theorem, for a certain class of spherical harmonics. Recall that \mathcal{H}_m is generated by the "special" spherical harmonics which I have defined in the Appendix 3,

In [D2] I defined the *order* of a spherical harmonic of the form of (5) as m_{d-2} , and the *type* as $m_0 - m_{d-2}$. Since the ultraspherical polynomials that appear in the expression (5) have degree $m_k + \frac{d-1-k}{2}$, and the m_k 's are decreasing, then we can also say that the order of Y is

the minimum order of these polynomials plus $\frac{1}{2}$, while the type of Y is the sum of the degrees of the polynomials in question. Note that the sum of the order and type of Y is m .

We can also define the order and type of a finite linear combination of spherical harmonics in \mathcal{L} . If $Y = \sum_{k=0}^M a_k Y_k(\omega)$, where $Y_k \in \mathcal{L}$ has degree m_k and order s_k , then the order of Y is $s = \min\{s_k\}$, and the type of Y is $\max\{m_k\} - s$.

To find the order of Y , a finite linear combination of spherical harmonics, one can also argue as follows: write Y in spherical coordinates, and observe that if the order of Y is $\geq s + \frac{1}{2}$, then $Y(\omega) = \left(\prod_{k=0}^{d-3} \sin \theta_{k+1}\right)^s Z_s(\theta_1 \dots \theta_{d-2})$, where $Z_s(\theta_1 \dots \theta_{d-2})$ is continuous in $[0, \pi]^{d-2}$. Therefore, the order of Y is $\bar{s} + \frac{1}{2}$, where \bar{s} is the maximum s for which $Z_s(\theta_1 \dots \theta_{d-2})$ is continuous.

The following Theorem shows that spherical harmonic of ‘‘high’’ order have better $L^p - L^q$ mapping properties than general spherical harmonics.

Theorem 4 *Let $Y(\omega)$ be a linear combination of spherical harmonics of degree $\leq m$. Suppose that $Y(\omega)$ has order s and type N . Then, for every $1 < p \leq 2$,*

$$\frac{\|Y\|_{L^2(S^{d-1})}}{\|Y\|_{L^p(S^{d-1})}} \leq c(N, s, p)^{\frac{N(d-2)}{2}} N^{\frac{1}{p}-\frac{1}{2}} \left(m + \frac{d-3}{p'}\right)^{\frac{d-2}{2}\left(\frac{1}{p}-\frac{1}{2}\right)}, \quad (10)$$

where

$$c(N, s, p) = C \max \left\{ 4e^{1+\frac{2}{p's}}, p' \left(1 + \frac{2N}{p's}\right) \right\},$$

and C does not depend on m , N and s .

Theorem 4 is a partial improvement of the Theorem that L. Grafakos and I proved in [DG]. Indeed, in [DG] we proved the inequality (10) for spherical harmonics in \mathcal{L}_m , (that is, not for linear combinations of spherical harmonics), with a constant C independent of N and s in place of $C(N, s, p)$, (see Section 6).

When $1 \leq p < \frac{2d}{d+2}$ and $q = 2$, then the exponent of m in Sogge’s theorem is $(d-1)\left(\frac{1}{p} - \frac{d}{2(d-1)}\right) > \frac{d-2}{2}\left(\frac{1}{p} - \frac{1}{2}\right)$. If we consider linear combinations of spherical harmonics of order s and type $N \leq \epsilon \ln s$, where $\epsilon > 0$ is a small constant, then, when s is sufficiently large, the function $c(N, s, p)$ in Theorem 4 is $\sim s^{\alpha\epsilon}$ for some positive constant α .

Therefore, when ϵ is sufficiently small, the upper bound in Theorem 4 is better than the upper bound in Sogge’s theorem.

A5. Carleman estimates and unique continuation

Let $P(D)$ be an elliptic operator of order $m \geq 2$ with constant coefficients. In 1939 T. Carleman discovered that a weighted inequality of the form of

$$\|e^{\tau\psi}u\|_{L^p(\mathbf{R}^d)} \leq C\|e^{\tau\psi}P(D)u\|_{L^q(\mathbf{R}^d)} \quad (11)$$

with $\frac{1}{p} - \frac{1}{q} = \frac{m}{d}$, which is valid for all $u \in C_0^\infty \mathbf{R}^d$, a suitable weight function ψ , values of $\tau > 0$ which are allowed to tend to $+\infty$, and a constant C which does not grow when $\tau \rightarrow \infty$, yields unique continuation properties of the differential inequality

$$|P(D)u(x)| \leq |V(x)||u(x)| \quad (12)$$

whenever $V \in L^r(\mathbf{R}^d)$, with $\frac{1}{r} = \frac{m}{d} = \frac{1}{p} - \frac{1}{q}$.

In this Appendix I prove a sample unique continuation theorem with the aid of a Carleman type inequality. The argument is borrowed from [KRS], (see also [Tesi]).

Let $u \in H^{m,p}(\mathbf{R}^d)$, be a solution of the differential inequality (12). Since $C_0^\infty(\mathbf{R}^d)$ is dense in $H^{m,p}(\mathbf{R}^d)$, we can see that u satisfies (11).

Suppose that u vanishes on one side of the hyperplane $\{x_1 = 0\}$. We show that if $u \equiv 0$ when $x_1 < 0$, then the Carleman type inequality

$$\|e^{-\tau x_1} u\|_{L^p(\mathbf{R}^d)} \leq C \|e^{-\tau x_1} P(D)u\|_{L^q(\mathbf{R}^d)} \quad (13)$$

implies that $u \equiv 0$ in \mathbf{R}^n . To do this we show first that there exists $r > 0$ which is such that $u \equiv 0$ when $x_1 \leq r$. Let $S_r = \{x : 0 \leq x_1 \leq r\}$, and let $r > 0$ be such that, if V is as in (16) and C is as in (13),

$$C \|V\|_{L^r(S_r)} \leq \frac{1}{2}.$$

By Hölder inequality, (13) and the inequality above, one can write the following string of inequalities.

$$\begin{aligned} \|e^{-\tau x_1} u\|_{L^q(S_r)} &\leq C \|e^{-\tau x_1} P(D)u\|_{L^p(\mathbf{R}^d)} \\ &\leq C \|e^{-\tau x_1} V u\|_{L^p(S_r)} + C \|e^{-\tau x_1} P(D)u\|_{L^p(\mathbf{R}^d/S_r)} \\ &\leq C \|V\|_{L^r(S_r)} \|e^{-\tau x_1} u\|_{L^q(S_r)} + C \|e^{-\tau x_1} P(D)u\|_{L^p(\mathbf{R}^d/S_r)} \\ &\leq \frac{1}{2} \|e^{-\tau x_1} u\|_{L^q(S_r)} + C \|e^{-\tau r} P(D)u\|_{L^p(\mathbf{R}^d/S_r)}. \end{aligned}$$

From the above follows that

$$\|e^{-\tau x_1} u\|_{L^q(S_r)} \leq 2C \|e^{-\tau r} P(D)u\|_{L^p(\mathbf{R}^d/S_r)},$$

and consequently,

$$\|e^{\tau(r-x_1)} u\|_{L^q(S_r)} \leq 2C \|P(D)u\|_{L^p(\mathbf{R}^d)}.$$

This inequality holds for every $\tau > 0$ and $r - x_1 > 0$, and so necessarily $u \equiv 0$ in S_r .

A6. Unique continuation and restriction theorems

Here I will explain why the restriction properties of the Fourier transform play a crucial role in the proof of $L^p - L^q$ Carleman estimates. Let $P(D)$ be an elliptic partial differential operator of order m with real constant coefficients. Assume for simplicity that $P(D)$ is homogeneous and has simple complex characteristics in the direction $(1, 0, \dots, 0)$ ²⁰.

We consider the following Carleman-type inequality

$$\|e^{\tau x_1} u\|_{L^p(\mathbf{R}^d)} \leq C \|e^{\tau x_1} P(D)u\|_{L^q(\mathbf{R}^d)}, \quad (14)$$

²⁰We recall that the complex characteristics of the operator $P(x, D)$ in the direction $\nu \in S^{n-1}$ are the complex roots of the polynomial $\tau \rightarrow P_m(x, \zeta + i\nu\tau)$, where $P_m(x, \zeta)$ denotes the principal symbol of $P(x, D)$.

²¹The Carleman estimate (14) implies that the nontrivial solutions of the differential inequality $|P(D)u(x)| \leq |V(x)|u(x)$ do not vanish identically on one side of the hyperplane $\{x_1 = 0\}$.

with $\frac{1}{p} + \frac{1}{q} = \frac{m}{n}$. Let

$$P_\tau(D) = e^{\tau x_1} P(D) e^{-\tau x_1} = P(D + i\tau e_1),$$

where we have set $e_1 = (1, 0, \dots, 0)$. Then (14) is equivalent to

$$\|u\|_{L^q(\mathbf{R}^d)} \leq C \|P(D + i\tau e_1)u\|_{L^p(\mathbf{R}^d)},$$

which in turn is (formally) equivalent to

$$\left\| \int_{\mathbf{R}^d} \frac{e^{i\langle \zeta, x \rangle}}{P(\zeta + i\tau e_1)} \widehat{u}(\zeta) d\zeta \right\|_{L^q(\mathbf{R}^d)} \leq C \|u\|_{L^p(\mathbf{R}^d)}. \quad (15)$$

After the change of variables $\zeta \rightarrow \tau\zeta$ we can assume $\tau = 1$. Let $\zeta = (\zeta_1, \zeta')$, with $\zeta_1 \in \mathbf{R}$ and $\zeta' \in \mathbf{R}^{d-1}$. Let $P(\zeta_1, \zeta')$ be the symbol of $P(D)$. After perhaps a change of coordinates we can let

$$P(\zeta_1, \zeta') = \zeta_1^m + \sum_{j=0}^{m-1} \zeta_1^j Q_j(\zeta'),$$

where the $Q_j(\zeta')$'s are homogeneous polynomials of degree $m - j$.

The roots $\lambda_1(\zeta') + i\mu_1(\zeta'), \dots, \lambda_m(\zeta') + i\mu_m(\zeta')$ of the polynomial $\zeta_1 \rightarrow P(\zeta_1, \zeta')$ are smooth homogeneous functions of degree 1 on $\mathbf{R}^{d-1} \setminus \{0\}$. Since $P(\zeta_1, \zeta')$ is real and elliptic, then the imaginary parts of the roots vanish only at the origin and, for every $j \leq m$, both $\lambda_j(\zeta') + i\mu_j(\zeta')$ and $\lambda_j(\zeta') - i\mu_j(\zeta')$ are roots. Therefore,

$$P(\zeta + ie_1) = P(\zeta_1 + i, \zeta') = \prod_{j=1}^{\frac{m}{2}} (\zeta_1 - \lambda_j(\zeta') + i(1 \pm i\mu_j(\zeta'))),$$

where $\mu_j(\zeta') > 0$ whenever $\zeta' \neq 0$. The “good factors” $\zeta_1 - \lambda_j(\zeta') + i(1 + \mu_j(\zeta'))$, $j = 1, 2, \dots, \frac{m}{2}$, never vanish, but the “bad” ones $\zeta_1 - \lambda_j(\zeta') + i(1 - \mu_j(\zeta'))$ may vanish on the (compact) manifold

$$S_j = \{\zeta : \lambda_j(\zeta') = \zeta_1, \mu_j(\zeta') = 1\}, \quad j = 1, \dots, \frac{m}{2}.$$

The manifold S_j has codimension 2, being the intersection of the “cone” $\{\lambda_j(\zeta') = \zeta_1\}$ and the “cylinder” $\{\mu_j(\zeta') = 1\}$. Since we have assumed that the characteristics of $P(D)$ are simple, the S_j 's do not intersect. The restriction theorems can be used to estimate the L^q norm on the

left hand side of (15) when ζ is in a neighborhood of the “bad” sets $\bigcup_{j=1}^{\frac{m}{2}} S_j$ and to prove the required inequalities.

6.1 A7. The Dirac operator, ([DO2]).

The Dirac operator is one of the most important system of partial differential equation in quantum mechanics. Let α_j , $j = 0, \dots, n$ be $N \times N$ symmetric matrices which form a basis of the Clifford algebra. Namely, they are anti commuting matrices which satisfy the following relations.

$$\alpha_j^* = \alpha_j, \quad \alpha_j^2 = I, \quad \alpha_j \alpha_k + \alpha_k \alpha_j = 0, \quad j \neq k,$$

where I is the $N \times N$ identity matrix, with $N = 2^{\lfloor \frac{n+1}{2} \rfloor}$, (see [Th]).

The (d -dimensional) Dirac operator

$$L = -i \sum_{j=1}^n \alpha_j \partial_{x_j}$$

is, roughly speaking, the “square root” of the Laplacian, in the sense that, (formally),

$$L^2 = -\Delta I.$$

Let Ω be a domain of \mathbf{R}^d containing the origin. In the joint paper [DO2] (with T. Okaji), we dealt with 2 different problems. The first one concerns the strong unique continuation property of the differential inequality

$$|Lu(x)| \leq |V(x)u(x)|, \quad x \in \Omega \quad (16)$$

and the other one concerns the strong unique continuation property of the differential equation

$$Lu(x) + V(x)u(x) = 0, \quad x \in \Omega. \quad (17)$$

In both cases we consider matrix potentials $V(x)$, (with a possible singularity at the origin), that satisfy ²²

$$|x|V(x) \in L_{loc}^\infty(\Omega)^{N^2}, \quad (18)$$

but in the case of the differential equation (17) and when the space dimension is equal to two or three it is possible to weaken the assumption on $V(x)$.

Here we only state our main results in the case $n \geq 2$.

Theorem 5 *Let $u(x) \in \{W_{loc}^{1,2}(\Omega)\}^N$ be a solution of the differential inequality (16). Suppose that $V(x)$ satisfies (18) and that $\sup_{x \in \Omega} |x| \|V(x)\| < \frac{1}{2}$. Then u is identically zero if it vanishes of infinite order at the origin.*

Theorem 6 *Let $w(x)$ be a spherically symmetric scalar function such that $|x|w(x) \in L^\infty(\mathbf{R}^d)$. Suppose that $V(x)$ satisfies (18) and*

$$\sup_{\{x \in \Omega\}} |x| \|V(x) - w(|x|)I\| < \frac{1}{2}.$$

If $u \in \{W_{loc}^{1,2}(\Omega)\}^N$ is a solution of the differential equation (17) and vanishes of infinite order at the origin, then u is identically zero in Ω .

Our results improve theorems of D. Jerison and V. Vogelsang. See [J], [V] and also [KY].

²²In view of the fact that $|\Delta Iu| = |LLu| \leq C|Lu| = C|\nabla u|$, we can translate the negative results for the Laplacean into negative results for the Dirac operator. Using an adaptation of a counterexample in [JK] we show that $|x|V(x)$ in (18) cannot be replaced by $|x|^{1+\epsilon}V(x)$ for any $\epsilon > 0$.

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