Solution Homework 3: MAA 4211 Spring 2002

4, page 37. (a) \(|x_n - a| \leq b_n\) for large \(n\) means that there exists \(N_0 \in \mathbb{N}\) such that
\[ |x_n - a| \leq b_n, \forall n \geq N_0. \]

Let \(\epsilon > 0\). Since \(b_n \to 0\), there exists \(N_1 \in \mathbb{N}, N_1 \geq N_0\), such that \(|b_n| < \epsilon, \forall n \geq N_1\).
Thus, for any \(n \geq N_1\), we have
\[ |x_n - a| \leq b_n < \epsilon. \]

(b) The conclusion that \(x_n\) converges to \(a\) remains true. The proof is as above, except \(N_1\) is chosen now so that \(N_1 \geq N_0\) and \(|b_n| < \epsilon/C, \forall n \geq N_1\).
\(\Box\)

8, page 37. \((\Rightarrow)\) Follows directly from Theorem 3.6.

\((\Leftarrow)\) We assume that any subsequence \(\{x_{n_k}\}_k\) of \(\{x_n\}_n\) converges to \(a\) and want to prove that \(\{x_n\}_n\) itself must converge to \(a\).
We consider a particular subsequence by taking \(n_k = k\) (notice that \(n_k < n_{k+1}\)). By assumption, the subsequence \(\{x_{n_k}\}_k\) converges to \(a\), but with our special choice \(x_{n_k} = x_k\), so the subsequence is nothing but the sequence itself.
\(\Box\)