Problems for Fermat’s and Euler’s Theorems

1. If \( p \) is prime, then, for all \( a \), \( a^p \equiv a \ (mod \ p) \).

2. What can you say about \( n \) and \( m \) if:
   (a) \( n^{96} \equiv m \ (mod \ 17) \)?
   (b) \( n^9 \equiv m \ (mod \ 19) \)?

3. (a) Show: if \( 7 \nmid n \), then \( 7 \mid (n^{12} - 1) \).
    (b) Show: \( n^{13} - n \) is divisible by \( 2, 3, 5, 7, 13 \), for all natural numbers \( n \).

4. (a) Find the remainder of the sum \( 1 + a + a^2 + a^3 + \ldots + a^9 \mod 11 \), for each number \( a < 11 \). Can you explain the outcome?
    (b) Make a theorem generalizing the statement from part (a) and prove your theorem.

5. Let \( N = \overline{111\ldots11} \), where \( N \) is a number in base 10, made up of \( p \) 1’s and where \( p \) is a prime other than 3. Show that \( N \equiv 1 \ (mod \ p) \).

6*. Let \( a, b \) be natural numbers such that \( gcd(a, b) = 1 \). Show that there exist numbers \( m, n \in \mathbb{N}^* \) such that
   \[ a^m + b^n \equiv 1 \ (mod \ ab). \]