Black and Scholes-Merton Model

I. Derivation of Black-Scholes Formula

Model Assumptions:
- \( dS = \mu S dt + \sigma S dZ \)
- No dividends
- Markets are friction-free:
  - No taxes
  - No Transaction costs
  - No short sales restrictions
  - Assets are divisible at will
  - Continuous trading
- \( r_{\text{borrowing}} = r_{\text{lending}} = \text{constant} \)

Derivative Pricing:

a. Let \( f(S,t) \) be the price of the derivative & use Ito’s lemma:

\[
\frac{df(S,t)}{dt} = \left[ \frac{\partial f(S,t)}{\partial t} + \mu S \frac{\partial f(S,t)}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f(S,t)}{\partial S^2} \right] dt + \left[ \frac{\partial f(S,t)}{\partial S} \sigma S \right] dZ
\]

b. Form a portfolio containing -1 unit of the derivative and
\( \Delta = \frac{\partial f(S,t)}{\partial S} \) shares of the underlying stock \( S \).

c. At time \( t \) (now), the portfolio value is:
\( \Pi = \Delta S - f(S,t) \)
d. Finally, compute the change in portfolio value:

\[ d\Pi(S,t) = \Delta dS - df(S,t) \]

\[ = \Delta \mu S dt + \Delta \sigma S dZ - \left\{ \frac{\partial f(S,t)}{\partial t} + \frac{\partial f(S,t)}{\partial S} \mu S + \frac{1}{2} \frac{\partial^2 f(S,t)}{\partial S^2} \sigma^2 S^2 \right\} dt - \left\{ \frac{\partial f(S,t)}{\partial S} \right\} dZ \]

\[ = -\left\{ \frac{\partial f(S,t)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(S,t)}{\partial S^2} \sigma^2 S^2 \right\} dt \]

Since there is no stochastic component, the portfolio is thus riskless. Therefore we must have: \( d\Pi(S,t) = r\Pi dt \) since the portfolio must earn exactly the risk-free rate (by non-arbitrage argument).

In other words,

\[ -\left\{ \frac{\partial f(S,t)}{\partial t} + \frac{1}{2} \frac{\partial^2 f(S,t)}{\partial S^2} \sigma^2 S^2 \right\} = r\left\{ \frac{\partial f(S,t)}{\partial S} S - f(S,t) \right\} \]

or:

\[ \frac{\partial f(S,t)}{\partial t} + \frac{\partial f(S,t)}{\partial S} rS + \frac{1}{2} \frac{\partial^2 f(S,t)}{\partial S^2} \sigma^2 S^2 - rf(S,t) = 0 \]

Note that since we did not specify what type of derivative was used, the equation above must be satisfied by EVERY derivative written on stock S.

Given an underlying process for S, what differentiates various derivatives written on that stock S is the type of initial/boundary conditions:

a. For a call option with exercise price K: \( f(S,T) = \max(0, S - K) \)

b. For a put option with exercise price K: \( f(S,T) = \max(0, K - S) \)

c. For a futures contract with delivery price K: \( f(S,T) = S - K \)
In the case of a European Call option, the solution is:

\[
C(S, K, t, T) = SN(d_1) - Ke^{-r(T-t)}N(d_2)
\]

- \(N(.)\) is the cumulative normal distribution
- \(d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\)
- \(d_2 = d_1 - \sigma\sqrt{T-t}\)

(Black-Scholes formula)

By Put-Call parity, the value of a put option can easily be obtained:

\[
P(S, K, t, T) = C(S, K, t, T) - S + Ke^{-r(T-t)}
\]

Finally, note that the non-arbitrage argument is very general and can be used to price any derivative.

II. Extensions of the Model

Relaxing the Log-Normal Assumption:

In reality, stock returns exhibit outliers too often to be consistent with the assumption made in part I.

1. Constant Elasticity of Variance Model:

   a. Instead of \(dS_t = \mu S_t dt + \sigma S_t dZ_t\), assume
   
   \[dS_t = \mu S_t dt + \sigma S_t^{1-\alpha} dZ_t\]

   where \(\alpha\) is a constant parameter.
b. This allows the instantaneous rate of return $dS/S$ to have a diffusion component that depends on the level of $S$, the stock price. Why?
For $\alpha > 0$, the volatility of $dS/S$ is INCREASING when $S$ is DECREASING, a convenient property since this is consistent with empirical observations.

c. Except for some special cases where $\alpha=0,1/2$, or 1, there is usually no closed-form solution.

2. **Jump-diffusion model:**

a. Jumps are often modeled as a Poisson process where the intensity or probability of a jump is determined by $\lambda$. The probability of a jump between time $t$ and $t+dt$ is thus $\lambda dt$ and the probability of a jump not occurring is finally $1-\lambda dt$.

b. If a jump happens and nothing else (pure jump), the stock price jumps from $S_t$ to $S_{t+\Delta t}=Y_t S_t$, so $\Delta S_t=(Y_t-1)S_t$ where $Y_t-1$ is the random jump size (percentage price change in decimal format). $Y_t$ is often lognormal (i.e. $Y_t=\text{exponential}(X)$ where $X$ is normally distributed) so that $Y_t-1$ has a range between -1 and infinity, consistent with a return (since percentage returns vary between -100% and infinity).

c. The jump-diffusion model is a diffusion model where a Poisson jump is added:

Replace $dS_t = \mu S_t dt + \sigma S_t dZ_t$ with the following:

$$dS_t = (\mu - \lambda \kappa)S_t dt + \sigma S_t dZ_t + S_t dq_t$$

where
• $\mu$ = expected return

• $q = \text{Poisson process generating the jumps, with } dq \text{ independent of } dZ \text{ and various jumps independent over time.}$

• $\kappa = E[Y - 1]$ = average jump size as a percentage (but in decimal format) of the stock price $S$.

• $\ln(Y)$ is normally distributed with variance $\delta^2$.

• Therefore:

$$dS_t = (\mu - \lambda \kappa)S_t dt + \sigma S_t dZ_t \quad \text{if the Poisson event does not occur}$$

$$dS_t = (\mu - \lambda \kappa)S_t dt + \sigma S_t dZ_t + (Y_t - 1)S_t \quad \text{if the Poisson event occurs}$$

In a Jump-diffusion setting, the European call option has for solution:

$$C(S, K, t, T) = \sum_{j=0}^{\infty} \frac{e^{-\lambda(t+\kappa)(T-t)} \{\lambda(1+\kappa)(T-t)\}^j}{j!} C_j$$

Where $C_j$ is the Black-Scholes Call price with

• volatility: $\sqrt{\sigma^2 + \frac{j \delta^2}{T-t}}$

• interest rate: $r - \lambda \kappa + \frac{j \ln(1+\kappa)}{T-t}$.
Relaxing the Constant Interest Rate Assumption:

In practice, unless the option is deep in-the-money and far away from maturity, the stochasticity of interest rates does not matter much.

Relaxing the Constant Volatility Assumption:

The volatility parameter can itself be made a stochastic process:

- Stock price: \( dS_t = \mu S_t \, dt + \sigma_{\text{volatility}} S_t \, dZ_t \)
- Variance \( \nu_t = \sigma_t^2 : \quad d\nu_t = \gamma \nu_t dt + \xi \nu_t dW_t \)

When the two Brownian motions \( Z \) and \( W \) are uncorrelated, the European call option price is given by:

\[
C = \int_{0}^{\infty} C(\bar{\sigma})g(\bar{\sigma}^2) d\bar{\sigma}^2,
\]

where:

- \( \bar{\sigma}^2 = \frac{1}{T-t} \int_{t}^{T} \sigma_u^2 \, du \) is the average variance and \( g(.) \) its probability density function.
- \( C \) is the Black-Scholes price evaluated at the average volatility \( \bar{\sigma} \).

For more details, see Hull & White (JF, 1987) or Wiggins (JF, 1987).

Note that the stochastic volatility features makes it more unlikely to obtain a closed-form solution.
III. Volatility Parameter Estimation

- Volatility is the only unobserved variable but is nevertheless an important one as option prices are sensitive to it.
- We thus need ways to attempt to measure it.

A. Standard Deviation of Returns

Recall that the Black-Scholes setup implies:

\[
\begin{align*}
\text{a. } & \ln(S_{t+\tau}/S_t) = \{\ln(S_{t+\tau}) - \ln(S_t)\} \sim N\left(\left\{\mu - \sigma^2/2\right\} \tau, \sigma^2 \tau\right) \\
\text{b. } & \text{Thus record all the stock prices } S_i \text{ at } \tau \text{ intervals, and by computing the log-returns between the intervals, calculate:} \\
\text{c. } & \tau \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} \left(\ln(S_{i+\tau}/S_i) - \bar{\ln}(S_{t+\tau}/S_t)\right)^2 \\
\end{align*}
\]

\[
\begin{align*}
\text{c. Since this yields the variance over a } \tau \text{ interval, the last step is to compute:} \\
\{\hat{\sigma}^2 \text{ over one year}\} = \frac{1}{\tau} \cdot \{\hat{\sigma}^2 \text{ over interval } \tau \text{ year}\} \\
\text{(In order to annualize the variance)}
\end{align*}
\]

Note that the more data the more precise the estimate. In order to increase the amount of data, one can either increase the time period or increase the frequency of data points recorded. However, too long a time period is not a good thing as volatility changes over time. Also, too high a frequency may pick up intraday microstructure perturbances that can bias the results.
B. Estimation of Stochastic Volatility Models

The previous section dealt with cases where the volatility is assumed to be constant. How do we estimate parameters in the stochastic case?

1. Stochastic Volatility Model:

\[
\begin{align*}
\text{d}S_t &= \mu S_t \text{d}t + \sigma_t S_t \text{d}Z_t \\
\text{d}\sigma_t &= \delta (\theta - \sigma_t) \text{d}t + \kappa \text{d}W_t
\end{align*}
\]

a. The volatility is said to follow an Ornstein-Uhlenbeck process, and it can be shown that the conditional distribution of \( \sigma_{t+\tau} \mid \sigma_t \) exhibits a normal distribution with the following characteristics:

- \( \mathbb{E}[\sigma_{t+\tau} \mid \sigma_t] = \theta + e^{-\delta\tau} (\sigma_t - \theta) \)
- \( \mathbb{V}[\sigma_{t+\tau} \mid \sigma_t] = \frac{\kappa^2}{2\delta} \left[ 1 - e^{-2\delta\tau} \right] \)

b. Notice that the distribution is known with certainty (the normal distribution is perfectly defined by two parameters only – the mean and the variance), and so if the volatility was observed we could perform a Maximum-Likelihood estimation.

c. A Quasi-Maximum Likelihood is also possible: by Taylor first-order expansion, we have \( e^{-\delta\tau} = 1 - \delta\tau + o(\tau) \), so \( \mathbb{E}[\sigma_{t+\tau} \mid \sigma_t] = \delta \tau \theta + (1 - \delta\tau)\sigma_t \) is an AR(1) model that can be estimated in the following manner:

- Estimate the time series of volatilities from the data.
• Then regress: \( \hat{\sigma}_{t+\tau} = \delta \hat{\theta} + (1 - \delta \tau) \hat{\sigma}_t + \nu_t \) to obtain estimates for \( \omega \) and \( \rho \). The variance of the residuals yield an estimate of \( \kappa^2 \tau \).

d. Generalized Method of Moments (GMM)

\[
\sigma_{t+1} - E[\sigma_{t+1} | \sigma_t] = \epsilon_{t+1}
\]

\[
\epsilon_{t+1} \approx N(0, V[\sigma_{t+1} | \sigma_t])
\]

\[
f_t(\Theta) = \begin{bmatrix}
\epsilon_{t+1} \\
\epsilon_{t+1} \times \sigma_t \\
(\epsilon^2_{t+1} - V[\sigma_{t+1} | \sigma_t]) \\
(\epsilon^2_{t+1} - V[\sigma_{t+1} | \sigma_t]) \times \sigma_t
\end{bmatrix}
\]

\[
g_T(\Theta) = \frac{1}{T} \sum_{t=1}^{T} f_t(\Theta)
\]

\[
J_T(\Theta) = g_T'(\Theta)W_T(\Theta)g_T(\Theta)
\]

The quadratic form \( J_T(\theta) \) is being minimized by the parameters. Note that the matrix \( W_T(\theta) \) is called the “weighing” matrix.

Regardless of the method used, once the parameters have been estimated they can then be plugged into the stochastic volatility option pricing model (closed-form formula if it exists) and be used to price options.