General Non-Arbitrage Model

I. Partial Differential Equation for Pricing
   A. Traded Underlying Security

      1. Dynamics of the Asset Given by:

         a. \[ dS = \mu(S,t)dt + \sigma(S,t)dZ \]
         b. The asset can be either a stock, or a currency, an index, a futures contract…

      2. Non-Arbitrage Pricing

         a. Markets are friction-free, as in Black-Scholes setting.
         b. Form a riskless portfolio which thus must earn the risk-free rate, as in the derivation of the Black-Scholes formula.
         c. PDE:

         \[
         \frac{\partial f(S,t)}{\partial t} + \frac{\partial f(S,t)}{\partial S} rS + \frac{1}{2} \frac{\partial^2 f(S,t)}{\partial S^2} \sigma^2(S,t)S^2 - rf(S,t) = 0
         \]

         d. The type of derivative will be determined by the specific final/initial/boundary conditions chosen.

      3. In Case the Asset Pays Dividends:

         a. Assume that the asset pays a continuous dividend yield \( \delta \), which means that between time \( t \) and \( t+dt \) the asset pays \( \delta Sdt \).
The riskless portfolio built will thus provide a return made up of a capital gain and a dividend, hence:

\[ \frac{d\Pi}{\text{capital gain}} + \Delta S dt = \frac{r\Pi}{\text{total return}} dt \]

b. The resulting PDE in this case is:

\[ \frac{\partial f (S, t)}{\partial t} + \frac{\partial f (S, t)}{\partial S} \{r - \delta \} S + \frac{1}{2} \frac{\partial^2 f (S, t)}{\partial S^2} \sigma^2 (S, t) S^2 - rf (S, t) = 0 \]

Notice that even though the drift of the underlying asset depends on the risk aversion of investors (high risk aversion implies a higher drift), the drift does not appear in the PDE. Hence derivative prices do NOT depend on the level of risk aversion of investors.

B. Non-Traded Underlying Asset or Factor
(such as interest rates, commodities, weather…)

1. Dynamics of the Factor Given by:

a. \( dF = m(F, t)F dt + s(F, t)F dZ \)

b. Consider two derivatives both written on factor F having prices \( f_1(F, t) \) and \( f_2(F, t) \) respectively.

By Ito’s lemma and using \( j=1 \) and \( j=2 \) for notation, we have:
\[
\frac{df_j(F,t)}{dt} = \left\{ \frac{\partial f_j(F,t)}{\partial t} + \frac{\partial f_j(F,t)}{\partial F} m(F,t) F + \frac{1}{2} \frac{\partial^2 f_j(F,t)}{\partial F^2} s^2(F,t) F^2 \right\} dt \\
+ \left\{ \frac{\partial f_j(F,t)}{\partial F} s(F,t) F \right\} dZ
\]

\[= \mu_j f_j dt + \sigma_j f_j dZ\]

c. In order to form a riskless portfolio, recall that we used to invest in the asset and the corresponding derivative. Here, the asset is a non-traded factor and it is therefore impossible to adopt the usual strategy. The new approach is to form a riskless portfolio by investing \((\sigma_2 f_2)\) in \(f_1\) and \((-\sigma_1 f_1)\) in \(f_2\).

d. The value of the resulting portfolio is thus given by:
\[
\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2
\]
while the change in value of the portfolio thus becomes:
\[
d\Pi = \left\{ \mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2 \right\} dt
\]

Since the random component is gone, the portfolio must therefore return the risk-free interest rate, and so:
\[
d\Pi = r \Pi dt
\]

2. **Market Price of Risk**

From above, we can conclude that:
\[
\left\{ \mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2 \right\} = r \Pi \\
= r \left\{ \sigma_2 f_2 f_1 - \sigma_1 f_1 f_2 \right\}
\]

And therefore that: \(\mu_1 \sigma_2 - \mu_2 \sigma_1 = r \sigma_2 - r \sigma_1\)

Which can finally be restated as: \(\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}\)
In other words, the ratio \( \frac{\mu_j - r}{\sigma_j} \) is the same for every derivative written on the same underlying factor F.

The ratio \( \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \equiv \lambda \) is called the market price of risk of factor F, and since both \( \mu \) and \( \sigma \) are functions of F and t, so is the market price of risk \( \lambda \).

3. The Pricing Partial Differential Equation

For any derivative of price \( f \), we must have the following:

\[
\mu - r = \lambda \sigma,
\]

where \( \mu(F,t) \) and \( \sigma(F,t) \) are determined by:

\[
\begin{align*}
\mu(F,t)f &= \frac{\partial f(F,t)}{\partial t} + \frac{\partial f(F,t)}{\partial F}m(F,t)F + \frac{1}{2} \frac{\partial^2 f(F,t)}{\partial F^2}s^2(F,t)F^2 \\
\sigma(F,t)f &= \frac{\partial f(F,t)}{\partial F}s(F,t)F
\end{align*}
\]

Therefore:

\[
\begin{align*}
\left\{ \frac{\partial f(F,t)}{\partial t} + \frac{\partial f(F,t)}{\partial F}m(F,t)F + \frac{1}{2} \frac{\partial^2 f(F,t)}{\partial F^2}s^2(F,t)F^2 \right\} - rf &= \lambda(F,t) \left\{ \frac{\partial f(F,t)}{\partial F}s(F,t)F \right\} \\
&\Downarrow \\
\frac{\partial f(F,t)}{\partial t} + \frac{\partial f(F,t)}{\partial F} \left\{ m(F,t) - \lambda(F,t)s(F,t) \right\}F + \frac{1}{2} \frac{\partial^2 f(F,t)}{\partial F^2}s^2(F,t)F^2 - rf &= 0
\end{align*}
\]

Eliminating the (F,t) notation for clarity yields:

\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \{m - \lambda s\}F} + \frac{1}{2} \frac{\partial^2 f}{\partial F^2}s^2F^2 - rf = 0
\]

Finally, to obtain the price of the derivative by solving this PDE, recall that final/initial/boundary conditions specific to the derivative must be stated.
4. **Case Where the Derivative Itself Pays Income**

Some derivatives, such as bonds (bonds are derivative securities since they derive their values from the value of an underlying factor, the interest rate), pay a stream of income (coupons in the case of bonds).

Assume that derivative $f_j(F,t)$ pays income $h_j(F,t)dt$ during interval of time $dt$. The portfolio $\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2$ remains riskless but we now have to take the income into account:

$$d\Pi + \underbrace{\left[(\sigma_2 f_2) h_1 - (\sigma_1 f_1) h_2\right]}_{\text{income}} dt = r\Pi dt$$

Since $d\Pi = \left\{ \mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2 \right\} dt$, we therefore have:

$$\frac{\mu_1 + (h_1/f_1) - r}{\sigma_1} = \frac{\mu_2 + (h_2/f_2) - r}{\sigma_2} \equiv \lambda$$

And the following PDE results:

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial F} \{m - \lambda s\} F + \frac{1}{2} \frac{\partial^2 f}{\partial F^2} s^2 F^2 + h - rf = 0$$
II. Equivalent Martingale Measure, Risk-Neutral Density (RND)

A. Expected Discounted Payoffs under RND

1. The Feynman-Kac formula says that

\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} a(S, t) + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} b^2(S, t) - c(S, t)f = 0
\]

with final condition \( f(S, T) = g(S) \) having for solution:

\[
f(S, t) = \mathbb{E}^Q \left[ g(S_T) \exp \left\{ - \int_t^T c(S_u, u) \, du \right\} \right]_{S_t = S}
\]

Note that the expectation sign is with respect to the density \( \rho^Q (S_T \mid S_t) \) obtained from the dynamics \( dS = a(S, t) \, dt + b(S, t) \, dZ_t^Q \). The density \( \rho^Q (S_T \mid S_t) \) is called the risk-neutral density or state price density.

If the pricing PDE happens to be

\[
\frac{\partial f}{\partial t} + \frac{\partial f}{\partial F} a(F, t) + \frac{1}{2} \frac{\partial^2 f}{\partial F^2} b^2(F, t) + h(F, t) - c(F, t)f = 0
\]

Then Feynman-Kac tells us that the solution is:

\[
f(F, t) = \mathbb{E}^Q \left[ g(F_T) \exp \left\{ - \int_t^T c(F_u, u) \, du \right\} + \int_t^T \exp \left\{ - \int_t^\tau c(F_u, u) \, du \right\} h(F_{\tau}, \tau) \, d\tau \right]_{F_t = F}
\]

Note that the main difference inside the expectation sign is that the sum of the discounted payments (h) was added.

2. Example with a Traded Underlying Asset

a. Actual Dynamics: \( dS = \mu(S, t) \, S \, dt + \sigma(S, t) \, S \, dZ \)
Risk-Neutral Dynamics: \( dS = \{r_t - \delta_t\} Sdt + \sigma(S, t) SdZ_t \)

The terms needed for the PDE are: \( a(S, t) = \{r_t - \delta_t\} S \), \( b(S, t) = \sigma(S, t) S \) and \( c(S, t) = r_t \) and so we get:

\[
(NT) \quad \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} \{r - \delta\} S + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 - rf = 0
\]

b. Notice that the actual drift rate is replaced by the risk-free rate (adjusted for the dividend yield) but that the volatility is unchanged. Also note that the final condition \( f(S, T) = g(S) \) is the derivative payoff function at terminal time \( T \).

c. If the interest rate is constant, the derivative price is:

\[
f(S, t) = E^Q \left[ g(S_T) \exp \left\{ -\int_t^T r u \right\} \right]_{S_t = S} = E^Q \left[ g(S_T) \exp \left\{ -r(T - t) \right\} \right]_{S_t = S} = e^{-r(T-t)} E^Q \left[ g(S_T) \right]_{S_t = S}
\]

d. And if the interest rate is stochastic:

\[
f(S, t) = E^Q \left[ g(S_T) \exp \left\{ -\int_t^T r \left( u \right) \right\} \right]_{S_t = S}
\]

3. **Example with a Non-Traded Underlying Factor**

a. Actual Dynamics: \( dF = m(F, t) Fdt + s(F, t) FdZ \)

Risk-Neutral Dynamics:

\[
dF = \left( m(F, t) - \lambda(F, t) s(F, t) \right) Fdt + s(F, t) FdZ^Q_t
\]

The terms needed for the PDE are: \( a(F, t) = \{m(F, t) - \lambda(F, t) s(F, t)\} F \), \( b(F, t) = s(F, t) F \) and \( c(F, t) = r \), and so we get:

\[
(NT) \quad \frac{\partial f}{\partial t} + \frac{\partial f}{\partial F} \{m - \lambda s\} F + \frac{1}{2} \frac{\partial^2 f}{\partial F^2} s^2 F^2 + h - rf = 0
\]
b. Notice that the actual drift rate \((mF)\) is replaced by the risk-free rate \((m-\lambda s)F\) but that the volatility is unchanged. Also note that the final condition \(f(F,T)=g(F)\) is the derivative payoff function at terminal time \(T\).

c. When the interest rate \(r\) is constant and when the derivative does not yield any cash payments \((h=0)\), the derivative price is:

\[
f(F,t) = E^Q \left[ g(F_t) \exp \left\{ -\int_t^T r u \, du \right\} \middle| F_i = F \right] \\
= E^Q \left[ g(F_t) \exp \left\{ -r(T-t) \right\} \middle| F_i = F \right] \\
= e^{-r(T-t)} E^Q \left[ g(F_t) \middle| F_i = F \right]
\]

d. In the most general case, however, we have:

\[
f(F,t) = E^Q \left[ g(F_t) \exp \left\{ -\int_t^T r_u \, du \right\} + \int_t^T \exp \left\{ -\int_t^\tau r_u \, du \right\} h(F_t,\tau) \, d\tau \middle| F_i = F \right]
\]

4. Girsanov Theorem

The Girsanov theorem reconciles the “actual dynamics” with the “risk-neutral dynamics”.

If the actual dynamics are described by:

\[
dS = \mu(S,t)Sdt + \sigma(S,t)SdZ
\]

The risk-neutral process can then be stated as:

\[
dS = (r - \delta)Sdt + \sigma(S,t)SdZ^Q
\]

The Girsanov states that if there is a process \(\lambda\), satisfying certain technical conditions (Novikov condition) and if there is a Brownian motion \(Z\) under the probability measure \(P\), then there is another probability measure \(Q\) (Equivalent Martingale Measure) that is equivalent to \(P\) in a way that the process
\[ dZ^0_t = dZ_t + \lambda_t \] is a Brownian motion under Q.

Applying Girsanov to the underlying asset’s process
\[ dS = \mu(S,t)Sdt + \sigma(S,t)SdZ, \]
we must have
\[ \lambda = \frac{\mu(S,t) - r + \delta}{\sigma(S,t)} \]
why?

Check:
\[ dZ^0_t = dZ_t + \lambda = dZ_t + \frac{\mu(S,t) - r + \delta}{\sigma(S,t)} \]
\[ S\sigma(S,t)dZ = S\sigma(S,t)dZ^0 - S[\mu(S,t) - r + \delta] \]
\[ dS = (r - \delta)Sdt + \sigma(S,t)SdZ^0 \]

The intuition for this is that a representative investor normally requires the instantaneous expected return \( \mu \) in order to hold the risky security.

However, in a fictitious “risk-neutral” world governed by a different probability measure - the equivalent Martingale measure - Q, that same investor would only require an expected rate of return equal to the risk-free rate.

Under some technical conditions, non-arbitrage is EQUIVALENT to the existence of the equivalent martingale measure Q, Harrison and Kreps (1979) and Harrison and Pliska (1981).
B. Example with Black-Scholes

1. **Apply Girsanov to:** \( dS = \mu S dt + \sigma S dZ \)

   a. The risk-neutral dynamics are: \( dS = r S dt + \sigma S dZ^Q \)

   b. The risk-neutral density \( \rho^Q(S_T | S_t) \) implied by the dynamics \( dS = r S dt + \sigma S dZ^Q \) is known and given by:

   \[
   \{ \ln(S_T) - \ln(S_t) \} \sim N \left( \left\{ r - \sigma^2/2 \right\} (T-t), \sigma^2 (T-t) \right)
   \]

   Therefore,

   \[
   \rho^Q \left( \ln(S_T) | \ln(S_t) \right) = \frac{1}{\sqrt{2\pi \sigma^2 (T-t)}} 
   \]

   \[
   \exp \left\{ - \frac{\left( \ln(S_T) - \ln(S_t) - \left\{ r - \sigma^2/2 \right\} (T-t) \right)^2}{2\sigma^2 (T-t)} \right\}
   \]

   By change of variable, we finally have:

   \[
   \rho^Q \left( S_T | S_t \right) = \frac{1}{S_t} \rho^Q \left( \ln(S_T) | \ln(S_t) \right)
   \]

2. **Pricing a European Call option**

   a. At expiration, the option payoff or value is:

   \[
   g(S) = \max(0, S - K)
   \]

   Therefore the option price is:

   \[
   C(S, K, r, t, T) = f(S, t) = e^{-r(T-t)} E^Q \left[ g(S_T) | S_t = S \right]
   \]

   \[
   = e^{-r(T-t)} E^Q \left[ \max(0, S_T - K) | S_t = S \right]
   \]

   \[
   = e^{-r(T-t)} \int_{0}^{\infty} \max(0, S_T - K) \rho^Q (S_T | S_t = S) dS_T
   \]

   \[
   = e^{-r(T-t)} \int_{K}^{\infty} (S_T - K) \rho^Q (S_T | S_t = S) dS_T
   \]