Equivalent Martingale Measure and Absence of Arbitrage

Recent methods of pricing derivatives written on assets rely on converting these assets’ prices into martingales, instead of relying on solving PDEs obtained from non-arbitrage arguments.

A. Probability Measure

1. Notation:

If a random variable \( x \) has \( f_x(x) \) for pdf, then

\[
P(x_0 - \Delta x/2 < x < x_0 + \Delta x/2) = \int_{x_0-\Delta x/2}^{x_0+\Delta x/2} f_x(x)dx
\]

One can make the interval centered around \( x_0 \) arbitrarily small, and as it approaches zero we denote

\[
P(x_0 - dx/2 < x < x_0 + dx/2) \quad \text{by} \quad dP(x_0)
\]

\( dP(x) \) is the probability that the random variable falls within an infinitesimally small interval \( dx \) centered around the value \( x \).

Summing all the \( dP(x) \) for the different possible values of \( x \) is thus written as:

\[
\int_{-\infty}^{+\infty} dP(x) \quad \text{with} \quad \int_{-\infty}^{+\infty} dP(x) = 1
\]
Risk-neutral pricing, as we are about to see, involves transforming the probability measure \(dP\) so that the mean of the random process \(x_t\) changes in a way that a risky asset can be treated as if it were risk-free.

To change the mean of a random variable, two things can be done: one can either change the possible values that the random variable can assume, or change the probabilities associated with these values. Since financial assets possess a range of values that cannot easily be changed, we will focus on modifying the probabilities.

2. Discrete Example (taken from Neftci, 2\textsuperscript{nd} edition):

Assume that the random variable \(x\) takes on the following possible values after we roll a 6-sided die:

\[
x = \begin{cases} 
10 & \text{if roll 1 or 2} \\
-3 & \text{if roll 3 or 4} \\
-1 & \text{if roll 5 or 6}
\end{cases}
\]

\[
\text{probabilities} = \begin{cases} 
1/6 & \\
1/6 & \\
1/3 & 
\end{cases} 
\]

\[
E(x) = (1/3)(10) + (1/3)(-3) + (1/3)(-1) = 2
\]

\[
\text{Var}(x) = (1/3)(10-2)^2 + (1/3)(-3-2)^2 + (1/3)(-1-2)^2
\]

\[
= 98/3
\]

Can we change the probabilities to obtain a different (pre-specified) mean without changing the variance?
Let us define $Q$ the new set of probabilities, if it exists. Let us now try to solve for the new set of probabilities.

a) Select the desired new mean, for example $E^Q(x) = 1$

b) Select the desired variance (unchanged): $\text{Var}^Q(x) = 98/3$

c) Condition: $Q(1 \text{ or } 2) + Q(3 \text{ or } 4) + Q(5 \text{ or } 6) = 1$

Solve the system and get:

- $Q(1 \text{ or } 2) = 122/429$
- $Q(3 \text{ or } 4) = 22/39$
- $Q(5 \text{ or } 6) = 5/33$

Check:

- $E^Q(x) = (122/429)(10) + (22/39)(-3) + (5/33)(-1) = 1$
- $\text{Var}^Q(x) = (122/429)(10-1)^2 + (22/39)(-3-1)^2$
  \[+ (5/33)(-1-1)^2 = 98/3\]

The desired result was thus obtained with the new probability measure $Q$.

**B. Girsanov Theorem**

1. **Preliminary Example:**

Suppose a random variable is such that $x_t \sim N(0,1)$. Its probability measure is therefore such that:

$$dP(x_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_t)^2} \, dx_t$$
Now define the following function: \( \xi(x_t) = e^{x_t \mu - \frac{1}{2} \sigma^2} \)

If we multiply \( \xi(x_t) \) by \( dP(x_t) \) we obtain a new probability because:

\[
dP(x_t) \xi(x_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_t)^2} e^{x_t \mu - \frac{1}{2} \sigma^2} dx_t = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_t - \mu)^2} dx_t
\]

Define \( dQ(x_t) = dP(x_t) \xi(x_t) \) and so we now have

\[
dQ(x_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x_t - \mu)^2} dx_t
\]

This is a normal distribution characterized by a mean of \( \mu \) and a variance of 1. The mean has been changed while the variance remained constant.

What this means is that for each small interval “dx” on the x-axis, we now have a new probability weight assigned.
2. Radon-Nikodym Derivative

Since $dQ(x_t) = dP(x_t)\xi(x_t)$ we then must have

$$\xi(x_t) = \frac{dQ(x_t)}{dP(x_t)}$$

This ratio can be seen as the “derivative” of the measure Q with respect to the measure P, or the density of Q with respect to P. This ratio is called the Radon-Nikodym derivative.

Recall that when deriving PDEs we have shown that derivative prices do not depend on the deterministic trend $\mu$ of the underlying asset’s process (only the risk-free rate $r$ appears in the PDEs) but that derivative prices do indeed depend on the volatility of the asset. The ability to change the mean $\mu$ to the riskless rate $r$ while keeping the volatility unchanged is thus very convenient for pricing options.

Note that as long as Q and P assign a non-zero probability to $dx$, we can go back and forth between the two measures:

$$dQ(x_t) = dP(x_t)\xi(x_t) \quad \text{and} \quad dP(x_t) = \xi(x_t)^{-1}dQ(x_t)$$

The two measures are thus called equivalent probability measures.
3. Girsanov Theorem Formalized

Define $\xi_t = e^{-\int_0^t \lambda_u du - \frac{1}{2} \int_0^t \lambda_u^2 du}$ for $t \in [0, T]$

where $\lambda_t$ is an $I_t$-measurable process (i.e., given the information set $I_t$, the value of $\lambda_t$ is known exactly) and $W_t$ a Brownian motion under the probability measure $P$.

If $E[e^{\int_0^T \lambda_u du}] < \infty$ (Novikov condition), then $\xi_t$ is a martingale with $E[\xi_T] = \xi_0 = 1$

and the process $W_t^Q = W_t + \int_0^t \lambda_u du$ is a Brownian motion under the probability measure $Q$ defined by $Q_T(A) = E^P[1_A \xi_T]$ (where $A$ is an event determined by $I_T$ and $1_A$ the indicator function of the event).

In other words, given a Brownian motion $W_t$, if we multiply the probability distribution of this process by $\xi_t$ we can obtain a new Brownian motion with probability distribution $Q$. The two processes are related through:

$$dW_t^Q = dW_t + \lambda_t dt$$

$Q$ is called the Equivalent Martingale Measure.
Note 1: To verify that \( Q_T(A) = E^P[1_A \xi_T] \), see that

\[
E^P[1_A \xi_T] = \int_A \xi_T \frac{dP}{dQ} dP = \int_A \xi_T dQ = Q(A)
\]

Note 2: Applying Ito’s lemma to \( \xi_t = e^{\int_0^t \lambda_u du} \)
yields:

\[
\frac{d\xi_t}{\xi_t} = -\lambda_t dW_t
\]

If \( \lambda_t \) is interpreted as the (time-dependent) risk premium, the transformation would then make all risky assets grow at the risk-free rate. For an underlying asset \( S \) such that

\[
dS = \mu S dt + \sigma S dW
\]

if we choose

\[
\lambda = \frac{\mu - r}{\sigma}
\]

then

\[
dS = \mu S dt + \sigma S[\int_0^t \lambda_u du] = \mu S dt + \sigma S\left[ \int_0^t \lambda_u du - \frac{1}{2} \lambda_u^2 du \right]
\]

yielding:

\[
dS = r S dt + \sigma S W^Q
\]

The intuition for this is that a representative investor normally requires the instantaneous expected return \( \mu \) in order to hold the risky security.

However, in a fictitious “risk-neutral” world governed by a different probability measure - the equivalent Martingale measure - \( Q \), that same investor would
only require an expected rate of return equal to the risk-free rate.

Under some technical conditions, non-arbitrage is EQUIVALENT to the existence of the equivalent martingale measure \( Q \), Harrison and Kreps (1979) and Harrison and Pliska (1981).

If the underlying asset is providing the investor with a continuous dividend yield \( \delta \), then we must choose

\[
\lambda = \frac{\mu(S,t) - r + \delta}{\sigma(S,t)}
\]

so that

\[
dS = \mu S dt + \sigma S [dW^0 - \lambda dt] = \mu S dt + \sigma S [dW^0 - \left( \frac{\mu - r + \delta}{\sigma} \right) dt]
\]

yielding:

\[
dS = (r - \delta) S dt + \sigma S dW^0
\]

4. State Price Deflator or Pricing Kernel

We know that by multiplying by \( \xi_t \), we transform the actual (objective) process into one with a risk-free drift provided that we carefully choose \( \lambda \) as the risk premium. We can go a step further and define for \( s > t \):

\[
\frac{\pi_s}{\pi_t} = e^{-\int_t^s r_u du} \cdot \xi_s
\]

\( \pi \) is the state price deflator or pricing kernel.
Since the Radon-Nikodym derivative only left us with a risk-free drift, now that $\pi$ is removing the risk-free rate we should observe that assets whose prices are multiplied by $\pi$ are actually martingales. As a check, for $s>t$:

$$E_t[\pi_s X_s] = E_t[\pi_t e^{\int_s^t \xi_s du} X_s] = \pi_t E_t^Q[e^{\int_s^t \xi_s du} X_s] = \pi_t X_t$$

The process $\pi_t X_t$ is therefore indeed a martingale.

**Note 1:** Applying Ito’s lemma to $\pi_t$ yields

$$\frac{d\pi_t}{\pi_t} = -r_t dt - \lambda_t dW_t$$

**Note 2:** The expression $E_t[\pi_s X_s] = \pi_t X_t$ can be stated as

$$X_t = E_t[\pi_t S]$$

Often written in asset pricing as: $X_t = E_t[m_s X_s]$

### C. Applications

1. **Lognormal Example:**

Suppose a random variable $S$ follows the lognormal distribution and is thus guided under the measure $P$ by:
\( S_t = S_0 e^{Y_t} \) where \( Y_t \sim N(\mu t, \sigma^2 t) \)

The new probability measure should have a different mean but identical variance. Let us thus define the probability measure \( Q \) by \( N(\rho t, \sigma^2 t) \) and attempt to identify the required mean.

For a constant risk-free rate and \( u < t \), we want:

\[
S_u = e^{-r(t-u)} E_u^Q [S_t] 
\]

Therefore we need:

\[
S_u = e^{-r(t-u)} E_u^Q [S_t] \quad \text{or} \quad S_u = e^{-r(t-u)} E_u^Q [S_u e^{Y_{t-u}}] 
\]

So we need:

\[
S_u = e^{-r(t-u)} E_u^Q [S_u e^{N[\rho(t-u), \sigma^2(t-u)]}] = e^{-r(t-u)} S_u e^{\rho(t-u)+(1/2)\sigma^2(t-u)} 
\]

(by property of the lognormal distribution)

Hence we must have:

\[
\rho = r - (1/2)\sigma^2 \quad \text{so that} \quad e^{-r(t-u)} S_u e^{\rho(t-u)+(1/2)\sigma^2(t-u)} = S_u 
\]

Note that if \( S_t = S_0 e^{Y_t} \) with \( Y_t \sim N(\mu t, \sigma^2 t) \), then by Ito’s lemma we have:

\[
\frac{dS}{S} = \left( \mu + \frac{1}{2} \sigma^2 \right) dt + \sigma dW 
\]

And so if we replace \( \mu \) by \( \rho = r - (1/2)\sigma^2 \), we get:

\[
\frac{dS}{S} = r dt + \sigma dW^Q 
\]

The instantaneous return is indeed risk-free.
2. **Black-Scholes European Call Option:**

Assume a process of the form: $dS = \mu S dt + \sigma S dZ$

a. The risk-neutral dynamics are: $dS = rS dt + \sigma S dZ^Q$

b. The risk-neutral density $\rho^Q(S_T | S_t)$ implied by the dynamics $dS = rS dt + \sigma S dZ^Q$ is known and given by:

$$\{\ln(S_T) - \ln(S_t)\} \sim N\left(\{r - \sigma^2/2\}(T-t), \sigma^2(T-t)\right)$$

Therefore,

$$\rho^Q\left(\ln(S_T) | \ln(S_t)\right) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \exp\left\{-\frac{\left(\ln(S_T) - \ln(S_t) - \left\{r - \sigma^2/2\right\}(T-t)\right)^2}{2\sigma^2(T-t)}\right\}$$

By change of variable, we finally have:

$$\rho^Q(S_T | S_t) = \frac{1}{S_T} \rho^Q\left(\ln(S_T) | \ln(S_t)\right)$$

**Pricing a European Call option**

At expiration, the option payoff or value is:

$$g(S) = \max(0, S - K)$$

Therefore the option price is:

$$C(S, K, r, t, T) \equiv f(S, t) = e^{-r(T-t)} E^Q\left[g(S_T) \mid S_t = S\right]$$

$$= e^{-r(T-t)} E^Q\left[\max(0, S_T - K) \mid S_t = S\right]$$

$$= e^{-r(T-t)} \int_0^{\infty} \max(0, S_T - K) \rho^Q(S_T | S_t = S) dS_T$$

$$= e^{-r(T-t)} \int_K^{\infty} (S_T - K) \rho^Q(S_T | S_t = S) dS_T$$