Numerical Methods in Option Pricing  
(Part II)

II. Binomial Trees

A. Cox-Ross-Rubinstein Tree (Black-Scholes setting)

a. We construct the tree to replicate the risk-neutral process of the underlying asset:

Where $f_n^m$ is the value of the derivative at node $(n,m)$ with $n$ the number of times the asset went up and $m$ representing the time period.
b. We can approximate the diffusion
\[ dS = rSdt + \sigma SdZ_t^Q \]
by:
\[
\begin{cases}
  u = e^{\sigma \sqrt{\Delta t}} \\
  d = e^{-\sigma \sqrt{\Delta t}} \\
  q = (a - d) / (u - d) \quad \text{where } a = e^{r\Delta t}
\end{cases}
\]

c. Starting from the final period, now use risk-neutral valuation backward, period by period:
\[
f_n^m = \frac{1}{1 + r\Delta t} \left[ q f_{n+1}^{m+1} + (1 - q) f_n^{m+1} \right]
\]

d. The value of the derivative in the final period is given by the payoff function \( g(.). \) In the case of the European call option, for instance, the payoff at time \( M\Delta t \) is:
\[ f_n^M = \max(0, S_u^n d^{M-n} - K), \quad n=0,1,\ldots,M. \]

e. To realistically approximate the derivative price, let \( \Delta t \) go to 0.

f. The advantage of such a tree is to have constant values for \( u, d, \) and \( q. \) They do not depend on the time period \( (m) \) nor do they depend on the stock level \( (n) \).
B. How to Estimate the “Greeks”

1. Delta (hedge ratio)
   a. We know that $\Delta = \frac{\partial f}{\partial S}$
   b. Delta can thus be estimated at node (n,m) by:

   $$\Delta^m_n = \frac{f^m_{n+1} - f^m_{n-1}}{S u^{n+1} d^{m-n-1} - S u^{n-1} d^{m-n+1}}$$

2. Other Hedge Ratios
   a. Since $\Gamma = \frac{\partial^2 f}{\partial S^2}$, $\Gamma^m_n = \frac{f^m_{n+1} - 2f^m_n + f^m_{n-1}}{[S u^{n+1} d^{m-n-1} - S u^{n-1} d^{m-n+1}]^2}$
   b. $\Theta = \frac{\partial f}{\partial t}$, $\Theta^m_n = \frac{f^m_{n+1} - f^m_{n-1}}{2\Delta t}$
   c. $\rho = \frac{\partial f}{\partial r}$. Need to construct two trees: one tree for $r$ and another for $r + \Delta r$. Estimate $\rho^m_n = \frac{f^m_n(r + \Delta r) - f^m_n(r)}{\Delta r}$.
   d. Vega = $\frac{\partial f}{\partial \sigma}$. Need to construct two trees: one tree for $\sigma$, and another for $\sigma + \Delta \sigma$. Estimate $\text{Vega}^m_n = \frac{f^m_n(\sigma + \Delta \sigma) - f^m_n(\sigma)}{\Delta \sigma}$. 
3. **Other Method to Compute Rho and Vega**

a. Instead of constructing two separate trees, recall that at the terminal period, we have: 

\[ f_n^M = \max(0, S_n^d M^n_n - K) \]

Thus at the period right before that, we have:

\[ f_n^m = \frac{1}{1 + r \Delta t} [q f_{n+1}^{m+1} + (1 - q) f_n^{m+1}] \]

- In the last period, we know that \( \partial f_n^M / \partial r = 0 \) since interest rates do not impact the final payoff. (Another way to view this is to notice that the final payoff is also the price of a derivative with zero time left to maturity, thus making discounting at \( r \) irrelevant). We can thus work backward and obtain earlier values of \( \partial f_n^m / \partial r \):

\[
\frac{\partial f_n^m}{\partial r} = \frac{\partial}{\partial r} \left( \frac{q}{1 + r \Delta t} f_{n+1}^{m+1} + \frac{1 - q}{1 + r \Delta t} f_n^{m+1} \right)
+ \left( \frac{q}{1 + r \Delta t} \right) \frac{\partial f_{n+1}^{m+1}}{\partial r}
+ \left( \frac{1 - q}{1 + r \Delta t} \right) \frac{\partial f_n^{m+1}}{\partial r}
\]

(Note that \( q \) depends on \( r \))

- Finally, the estimate of the options’ rho is the value of the expression above at the first node (\( m=n=0 \)), since rho = \( \partial t_0^0 / \partial r \).

b. Notice that the same methodology can be employed to compute Vega.
C. How to Adjust for Payments by the Underlying Asset (Options on: Index, Futures, and Currencies)

a. Case of an Index Option (ex: S&P 500 options):
   Replace r in $a = e^{r\Delta t}$ by $r - \delta$ ($\delta$=dividend yield) in the formula for the risk-neutral probability $q$.
   So $a = e^{(r-\delta)\Delta t}$.

b. Case of an Option on Futures:
   Unlike a stock, an index, or a currency, futures contracts are simply agreements between two parties and thus do not require any payment upfront (ignoring margin requirements and such). Since it costs nothing to enter such a contract, the risk-neutral dynamics of the futures $F$ must be: $dF* = \sigma F* dZ$ (no drift, or $\delta = r$ in the expressions above). So $a = 1$.

c. Case of a Currency Option:
   Example: 1-year foreign currency European call option on the British Pound. The US risk-free rate is denoted by $r_{US}$ and the UK risk-free rate is denoted by $r_{UK}$.

   • The underlying factor is the exchange rate and its dynamics are given by: $de^* = (r_{US} - r_{UK})e^* dt + se^* dW$.

   • It is mathematically the same thing as having a dividend payout, with $r=r_{US}$ and $\delta=r_{UK}$. Thus we have $a = e^{(r_{US}-r_{UK})\Delta t}$ here.

d. Finally notice that the discounting still takes place at the same rate $r$, so $C = [qC_u + (1-q)C_d]/[1 + r\Delta t]$. The only difference is that we have adjusted $q$ to reflect $r-\delta$ in “a”.
III. Numerical Methods for PDEs

A. Types of PDEs and Terminology

a. Example of a simple Pricing PDE we have seen for a traded asset:

\[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} (r - \delta) S + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 - rf = 0 \]

b. Example of a simple Pricing PDE we have seen in the case of a non-traded factor:

\[ \frac{\partial f}{\partial t} + \frac{\partial f}{\partial F} (m - \lambda s) F + \frac{1}{2} \frac{\partial^2 f}{\partial F^2} s^2 F^2 + h - rf = 0 \]

c. General form for Second Order Linear PDEs:

\[ a(S,t) \frac{\partial^2 f}{\partial S^2} + b(S,t) \frac{\partial^2 f}{\partial S \partial t} + c(S,t) \frac{\partial^2 f}{\partial t^2} \]

\[ + d(S,t) \frac{\partial f}{\partial S} + e(S,t) \frac{\partial f}{\partial t} + g(S,t) f + h(S,t) = 0 \]

The discriminant \( \Delta \) is \( b^2 - 4ac \)

- If \( \Delta \) is positive, the equation is said to be hyperbolic.
- If \( \Delta \) is negative, the equation is said to be elliptic (never happens in finance).
- If \( \Delta = 0 \), the equation is said to be parabolic (the type we encounter in finance where \( b = c = 0 \)).
- Notice that this way of classifying is local: an equation may be elliptic at some \((S,t)\) and parabolic at others.
d. Backward vs. Forward PDEs:

\[
\frac{\partial f}{\partial t} + a(S,t) \frac{\partial^2 f}{\partial S^2} + d(S,t) \frac{\partial f}{\partial S} + g(S,t) f + h(S,t) = 0
\]

- If \(a > 0\), the equation is backward.
- If \(a < 0\), the equation is forward.

B. General Tricks to Find a Closed-Form Solution

Before resorting to numerical methods, it is worth first checking whether a closed-form solution exists. Looking for a PDE of the same form in a book referencing PDEs with known solutions at the Math library is generally a good way to go, but here are a few general tips that may help you solve the equation on your own.

a. Apply successive changes of variables to attempt to transform the original equation into the heat equation that has a known solution. Example: \(x = \log(S)\) or \(x = \log(S/K)\), \(\tau = (T-t)/(\sigma^2/2)\).

b. Decrease the number of parameters to get the equation to a simpler form, for example: \(\kappa_1 = r/(\sigma^2/2)\) and/or \(\kappa_1 = (r-\delta)/(\sigma^2/2)\).

c. Try solutions in the form \(V(x,\tau) = e^{\alpha x + \beta \tau} U(x, \tau)\), or \(V(x,\tau) = \tau^\gamma W(x/\tau^\delta)\) and choose the parameters \(\alpha\) and \(\beta\) or \(\gamma\) and \(\delta\) to simplify the equation.
C. PDEs Numerical Approximation

1. First Derivatives

a. By definition, we have:

\[ \frac{\partial f(S,t)}{\partial t} = \lim_{\Delta t \to 0} \frac{f(S, t + \Delta t) - f(S, t)}{\Delta t} \]
\[ = \lim_{\Delta t \to 0} \frac{f(S, t) - f(S, t - \Delta t)}{\Delta t} \]
\[ = \lim_{\Delta t \to 0} \frac{f(S, t + \Delta t) - f(S, t - \Delta t)}{2\Delta t} \]

(For \( S \), we have: \[ \frac{\partial f(S, t)}{\partial S} = \lim_{\Delta S \to 0} \frac{f(S + \Delta S, t) - f(S, t)}{\Delta S} \] etc.)

b. Thus we can approximate the derivative by calculating the right-hand side of the equations above at a small \( \Delta t \):

- **Forward Difference:**

\[ \frac{\partial f(S, t)}{\partial t} \approx \frac{f(S, t + \Delta t) - f(S, t)}{\Delta t} \]

- **Backward Difference:**

\[ \frac{\partial f(S, t)}{\partial t} \approx \frac{f(S, t) - f(S, t - \Delta t)}{\Delta t} \]

- **Central Difference:**

\[ \frac{\partial f(S, t)}{\partial t} \approx \frac{f(S, t + \Delta t) - f(S, t - \Delta t)}{2\Delta t} \]
2. Second Derivatives

a. By definition, we have:

\[
\frac{\partial^2 f(S, t)}{\partial S^2} = \lim_{\Delta S \to 0} \frac{\partial f(S + \Delta S, t) - \partial f(S, t)}{\partial S} \frac{\partial f(S, t)}{\partial S} \Delta S
\]

\[
= \lim_{\Delta S \to 0} \frac{\partial f(S, t) - \partial f(S - \Delta S, t)}{\partial S} \frac{\partial f(S - \Delta S, t)}{\partial S} \Delta S
\]

\[
= \lim_{\Delta S \to 0} \frac{\partial f(S + \Delta S, t) - \partial f(S - \Delta S, t)}{2 \Delta S}
\]

b. Using the symmetric central difference, we obtain:

\[
\frac{\partial^2 f(S, t)}{\partial S^2} \approx \frac{\partial f(S + \Delta S, t) - \partial f(S, t)}{\partial S} \frac{\partial f(S, t)}{\partial S} \Delta S
\]

\[
\approx \left\{ \frac{f(S + \Delta S, t) - f(S, t)}{\Delta S} \right\} \left\{ \frac{f(S, t) - f(S - \Delta S, t)}{\Delta S} \right\}
\]

\[
\approx \frac{f(S + \Delta S, t) - 2f(S, t) + f(S - \Delta S, t)}{(\Delta S)^2}
\]

(It can be shown that the central approximation is an order of magnitude more precise than either the forward or backward difference).
D. Finite Difference Grid

1. Construct an Equally-Spaced Grid

a. The PDE that needs to be solved is:

\[ \frac{\partial f}{\partial t} + a \frac{\partial f}{\partial S} + b \frac{\partial^2 f}{\partial S^2} + h - rf = 0 \]

for \( S < S < \overline{S} \) and \( t > 0 \)

Subject to:

- \( f(S,0) = f_0(S) \) [initial condition]

(Notice that \( a, b, \) and \( h \) are allowed to be functions of \( S \) and \( t \)).

In option pricing, we usually have a terminal condition instead of an initial one. We can switch between the two with a change of variable from \( t \) to \( \tau = T - t \). This change of variable therefore changes \( \frac{\partial f}{\partial t} \) to \( -\frac{\partial f}{\partial \tau} \).

The PDE therefore becomes:

\[ -\frac{\partial f}{\partial \tau} + a \frac{\partial f}{\partial S} + b \frac{\partial^2 f}{\partial S^2} + h - rf = 0 \]
b. **Construct a grid in time and state (state = stock price)**

The goal is to solve for the various values of the derivative price $f$ on the various points of the grid, where we define $f_{n}^{m} = f(n\Delta S, m\Delta t)$. Note that we will interpolate between grid points, and that time on the grid is defined as time-to-expiration.
2. **Implementation**

There are different types of finite-difference approximations. The specific type we chose to implement for the first and second-order derivatives that appear in the Partial Differential Equation will determine the nature of the algorithm we use.

   a. Explicit

   b. Implicit
      - Fully Implicit
      - Crank-Nicolson