5.7 The Poisson Probability Distribution

In general, we define a random process to be a physical process which is completely or partially controlled by some chance mechanism. As an example, the number of customers that visit a store on a given day, the number of calls received at a switchboard during a given period of time, .etc. Note that all of these processes are time dependent - that is the events do or do not take place at regular intervals of time or through continuous time intervals.

In a Poisson Process, we count the number of occurrences of an event in continuous time intervals.

The random variable of interest then is:

X = # of occurrences in a time interval.

Properties of a Poisson Experiment

1) The probability of the occurrence of a single event in a very small interval of length Δt is given by $\alpha\Delta t$.

2) The probability of the occurrence of more than one event in such a small interval is negligible.

3) Event occurrences in non overlapping time occurrences are independent.

The Poisson Probability Distribution

The probability of the occurrence of x events in a time interval of length T is given by

p(y) = $\frac{\lambda^x e^{-\lambda}}{x!}, \; x = 0,1,2,...$

$\lambda = \alpha T.$

Note that $\lambda = E(X)$, is the mean number of occurrences in the time T, so that α = average # of occurrences in a single unit of time.

Mean, Variance and m.g.f of a Poisson Random Variable.

Mean = $E(X) = \lambda$

Variance = $\sigma^2 = \lambda$

$M_X(t) = e^{\lambda(e^t - 1)}$

Remember that the above is the mean and variance in an interval of length T

Examples:

1) Flaws in a material occur on an average of λ = one per yard. Find the following:

a) Probability of getting 3 flaws in a one yard piece.

p(3) = ______________ = 0.063.

b) Probability of getting at most three flaws in a one yard piece.

$P(Y \leq 3)$ = _________________

c) Probability of getting 10 flaws in a 6 yard piece.

What is the mean number of flaws in a 6 yard piece? _________

Hence, p(6) = _____________ = 0.041.

2) A Geiger counter records 40 counts per minute on an average when in the neighborhood of a certain weakly radioactive substance. Find the probability that there will be:

a) 2 counts in a six second period.
b) at least three counts in a ten second period.
c) at most 2 counts in a 10 second period.

What is α, the average number of counts per second?

Hence, the average number of counts in a 6 second period is ___________ and in a 10 second period is _____________.

Now in (a), p(2) = __________________ = 0.146

b) $P(Y \geq 3)$ = ___________________ = ______________ = 0.96.

c) ____________ = ________________.

Chapter 6 – Special Probability Densities

6.2 - The Uniform Distribution

A continuous random variable X takes on the Uniform Distribution on the interval (α,β) if X can take on any value on the interval and if all these values are equally likely: The p.d.f of X is given by:

$$f(x) = \begin{cases} \dfrac{1}{\beta-\alpha}, & \alpha \le x \le \beta \\ 0, \text{otherwise} \end{cases}$$

The graph of f(x) is:

The expected values are:

$$E(X) = \frac{\beta+\alpha}{2}, \quad Var(X) = \frac{(\beta-\alpha)^2}{12}$$

The c.d.f is given by:

$$F(x) = \begin{cases} 0, & x \le \alpha \\ \dfrac{x-\alpha}{\beta-\alpha}, & \alpha < x < \beta \\ 1, & x \ge \beta \end{cases}$$

Physical examples:

Some direct examples – but mainly used for its simplicity

6.3 The Exponential Distribution:

Derivation: The time between failures of a Poisson process

$P(L \le t) = 1 - P(L > t) = 1 - P(0 \text{ failures in } (0,t))$

The density function of this distribution is given by:

$$f(x) = \begin{cases} \frac{1}{\theta}\exp(-x/\theta), & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

θ is called the scale parameter and has to be positive.
The graph of the density function is:

The mgf of the distribution is:

$$M_X(t) = \frac{1}{1-\theta t}$$

The expected values are:

$E(X) = \mu = \theta, \quad Var(X) = \sigma^2 = \theta^2$

Physical Examples:

1) Waiting time between successive breakdowns of a system.
2) Waiting time between arrivals of planes at an airport.
3) The time a watch will run without having to be reset.

6.3 - The Gamma Distribution

The Gamma Distribution has the exponential as a special case.

The general form of the p.d.f of the Gamma Random Variable, X, is given as

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} \exp(-x/\beta), \; x \geq 0 \\ 0, \text{ otherwise.} \end{cases}$$

α and β are parameters of the distribution and have to be positive.

$\Gamma(\alpha)$ is called the gamma function and is given by:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} \exp(-x) dx$$

For a positive integer α, we can show that $\Gamma(\alpha) = (\alpha-1)!$

$E(X) = \mu = \alpha\beta$ and $Var(X) = \sigma^2 = \alpha\beta^2$.

We look at a Poisson Process and consider the waiting time to the αth arrival.
Let us look at the derivation: think of α iid exponential random with mean β:

The gamma random variable is the sum of these variables
(a) Waiting time to the 10th breakdown of a system.

(b) Waiting time between the arrival of the 1st and the 10th plane, between the 11th and the 20th plane and so on...

6.4 - The Beta Distribution

For a Binomial Distribution, when n gets very large, the proportion of successes has a Beta Distribution.

A random variable has a Beta Distribution with parameters α and β if the density function of X has the following form:

$$f(x) = \begin{cases} \dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}, \; 0 < x < 1 \\ \qquad\qquad\qquad\qquad\qquad\qquad \alpha > 0, \beta > 0 \\ 0, \text{ otherwise.} \end{cases}$$

$$\mu = \frac{\alpha}{\alpha+\beta}, \qquad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Physical Examples:

1) Proportion of successful free throws during the entire career of a basketball player.

2) Proportion of defective transistors produced annually at a factory .....

4.8 - The Weibull Distribution

Lifetimes of products have this distribution. A random variable X has the Weibull Distribution with parameters γ and θ if the density function of X is given by :

$$f(x) = \begin{cases} \dfrac{\gamma}{\theta} x^{\gamma-1} \exp(-x^{\gamma} / \theta), & x > 0, \gamma > 0, \theta > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$E(X) = \theta^{1/\gamma}\, \Gamma(1 + 1/\gamma) \text{ and } \sigma^2 = \theta^{2/\gamma} \{\Gamma(1 + 2/\gamma) - (\Gamma(1 + 1/\gamma))^2\}$$

Vacuum tube failures, ball bearing failures, life of semiconductor devices are all described by the Weibull Distribution.