MAC 2311 Feb 8, 2006
Exam I and Key Prof. S. Hudson

Show all your work and reasoning for maximum credit. If you continue your work on another page, be sure to leave a note. Do not use a calculator, book, or any personal paper. You may ask for extra paper; but please return it with your exam.

1) [20pt] Short answer problems. You may answer with '$+\infty$' or '$-\infty$' but not with 'd.n.e'.

a) Solve for x, given that $\frac{1}{2-x} < 1$.

b) $\lim_{x\to 1} \frac{x-1}{\sqrt{x}-1} =$

c) $\lim_{y\to 6^+} \frac{y+6}{y^2-36} =$

d) $\lim_{x\to +\infty} \frac{e^x+e^{-x}}{e^x-e^{-x}} =$

2) [20pt] Same instructions as problem 1.

a) $\lim_{x\to 0} \frac{\tan(5x)}{\sin(3x)\cos(2x)} =$

b) $\lim_{\theta\to 0} \frac{\sin^2(\theta)}{\theta} =$

c) $\lim_{x\to +\infty}(1 + \frac{1}{2x})^x =$ Hint: try a substitution.

d) $\lim_{x\to +\infty} \frac{3x^3+2x^2+x}{x^3+2x^2+3x+4} =$

3) (15pts) Answer True or False. You do not have to explain.

a) $\tan(x)$ is continuous on $[\pi/4, \pi]$.

b) $\frac{\cos(x)}{\ln(x)}$ is continuous on $[\pi/4, \pi]$.

c) $\lim_{x\to 0} \frac{\sin(x^3)}{x^3} = 1$.

d) $\forall \epsilon > 2, \exists \delta > 3, \delta < \epsilon$.

e) $\forall \epsilon > 4, \exists \delta > 3, \delta < \epsilon$.

4) [10pt] Sketch the curve by eliminating the parameter: $x = \sqrt{t}$ and $y = 2t + 1$.

5) [10pt] Approximate the solution to $x^3+x^2-x+1 = 0$ within 0.1, with some explanation of your reasoning. You can use the data below instead of a calculator (a little arithmetic and organization is left for you).

x	$x^3 + x^2$
1	2
-1	0
-2	-4
-1.8	-2.592
-1.6	-1.536
-1.4	-0.784

6) [15pt] Refer to the figure below. You can assume $0 < \theta < \pi/2$ as it is drawn. You can also use the Squeeze theorem or Intermediate Value Theorem in this problem if you need them, but do not make the assumption that $\sin(x)$ is continuous.

6a) Find a formula for the area of the wedge OPQ

6b) Use the picture (you may need to draw another line) to explain that $\sin(\theta) < \theta$.

6c) Prove that $\lim_{x \to 0} \sin x = 0$ [See the instructions for problem 6). You can also use your answers to 6a) or 6b) here].

7) [10pts] Choose ONE of the problems below to do. Remember to use enough words and sentences - not just formulas.

a) Show that $\lim_{x \to 0} \sin(x)/x = 1$

b) Show that $\lim_{x \to 3} 4x - 1 = 11$ using the definition of limit.

Bonus [5pts]: Use the definition of limit to prove that $\lim_{x \to 2} 1/x = 1/2$. This should be similar to the limit problem with x^2 we did in class (though the algebra will be a little different). It may be difficult, so don't try it until you have finished the rest of the exam. You can assume that $\delta < 1$ if you want. As usual, you can answer on the back, but if you do, leave me a note here.

Remarks + Answers: The average of the top 20 scores was 62/100. The scores were pretty good on problems 1, 4 and 7; but pretty low on problems 5 and 6. As always, check my addition and let me know if you spot a mistake. Note that the four parts of problem 1 have been added together, with the sum placed in the margin - so don't add that in twice. Same for problem 2abcd and 6ab (but 6c is separate). I expect to adjust the scale downwards about 5 points for this exam - we'll discuss this in class.

1a) This is simpler version of a problem from the PreTest. Again there are several ways to solve it. I suggest setting $2 - x = 0$ and $1/(2 - x) = 1$ to get the critical numbers 1 and 2. Then a little testing shows $x < 1$ *or* $x > 2$ (don't omit the 'or'). You can also write the answer as $x \in (-\infty, 1) \cup (2, \infty)$.

1b) 2 (Multiply by the conjugate).

1c) $+\infty$ (Factor the den).

1d) 1 (similar to 2d)

2a) 5/3

2b) 0

2c) $e^{1/2}$ (Set $h = 2x$ and use Ch 2.3, Example 3. Maybe this problem was too hard without similar HW exercises. See 2.3.65).

2d) 3 (ignore the lower-order terms, or divide top/bottom by x^3).

3) FFTFT

4) $y = 2x^2 + 1$ with $x \geq 0$. This forms the right half of a parabola with vertex at (0,1). Your sketch need not be perfect, but it should curve upwards, and pass through (1,3).

5) Similar to Ch 2.5 Example 5 [and HW 2.5.43]. Plug in lots of x values until you can make $f(x_1)$ negative and $f(x_2)$ positive, with $x_1 \approx x_2$ (then quote the Intermediate Value Theorem). With the data provided, you can easily check that $f(-2) = -1 < 0$ and $f(-1.8) = -2.592 + 1.8 + 1 > 0$. So, there is a solution in the interval $(-2, -1.8)$. So, one good answer is $x = -1.9$, which is within 0.1 of the exact solution.

If you gave an answer like 1.85 (OK, except it's missing a minus sign), then I probably gave some credit. But if I couldn't follow your work, then I couldn't give you much. If you mentioned the IVT, I gave at least a few points.

6a) $A = \theta/2$. This famous formula was one small step of the $sin(x)/x$ proof. But the angle in the picture is labelled θ this time, not x. So, don't write $A = x/2$.

6b) Draw a vertical line segment from P down to the x-axis. It's length is $\sin(\theta)$. It is shorter than the line segment PQ, which is shorter than the arc PQ, which has length θ.

Some people went into the proof of the $\sin(x)/x$ theorem, but it doesn't quite work. You can get to $\sin(x)\cos(x) < x$, but not to $\sin(x) < x$ that way. For my own convenience, I put the sum of your 6a and 6b grades into the margin of page 3, then graded 6c separately on page 4. Each part was worth 5 points.

6c) Apply the Squeeze Theorem to $0 < \sin(x) < x$ (as $x \to 0$).

7) Most people chose b) and did OK. The grades were usually better when people used the same style as in my lectures. The textbook method is also correct, but it is harder to explain the logic properly.

Bonus: Nobody got this one quite right, but several people got partial credit. Some keys steps: $|1/x - 1/2| = |2-x|/|2x|$. If $|x-2| < \delta < 1$ then $1 < x < 3$ and $1/|2x| = 1/2x < 1/2$. So, set $\delta \leq 2\epsilon$ [with $\delta < 1$]. Get $|2 - x|/|2x| < (2\epsilon)(1/2) = \epsilon$.