

1) (10 pts) a) Write out a limit from Ch 6.9 which is equal to  $e$  (there are three good choices).

1b) Use a substitution to compute  $\lim_{x \rightarrow +\infty} (1 + \frac{3}{x})^x$ . Show all your work clearly!

2) (10 pts) A cone-shaped water tank is 12 ft tall, has a radius of 4 ft at the top, and is contains water, 8 feet deep (62 lbs per cu.ft.). Use an integral to find the work required to pump the water out the top of the tank. [Compute the integral, but you do not have to multiply out the constants].

3) (35 pts) Compute each:

a)  $\int x \ln(x) dx$

b)  $\int \sin^3 x \cos^2 x dx =$

c)  $\int 5^x dx =$

d)  $\int \sin^{-1}(x) dx$

e)  $\int_{-1}^1 \frac{1}{1+x^2} dx$

4) (10pts) Choose ONE.

a) Find the arc length of  $y = x^{2/3}$  from  $x = 1$  to  $x = 8$ .

b) Find the surface area, when  $y = \sqrt{4 - x^2}$ ,  $-1 \leq x \leq 1$ , is revolved around the  $x$ -axis.  
Hint:  $S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$ .

5) (10pt) Find the volume, when the region bounded by these curves is revolved around the  $y$  axis:  $y = \sqrt{x}$ ,  $x = 4$ ,  $y = 0$ .

6) (15 pts) Answer True or False for each:

The volume of a right cylinder is  $V = Ah$ .

Distance travelled =  $\int_{t_0}^{t_1} v(t) dt$ .

If two water tanks both contain the same volume of water, then the work required to empty them is the same.

The region under  $y = 3 + (x - 1)^2$  over  $[0, 2]$ , is revolved around the  $x$ -axis. Then the best way to find the volume is the shell method.

The domain of  $\exp(x)$  is the same as the domain of  $\tan^{-1}(x)$ .

7) (10pts) Choose ONE, and explain thoroughly:

a) Explain (prove) the integral formula used in the Washer Method. Include: a picture, a limit, a sum, the volume of a washer, and plenty of words.

b) Prove the formula for  $\ln(ab)$  using the integral definition.

c) Derive the reduction formula for  $\int \sin^n x \, dx$ .

BONUS: (5 pts) Find  $\int \cos^3 x \, dx$  without using a reduction formula.

**Remarks and Answers:** The average was approx 58/100, based on scores over 45/100. The unofficial scale is: A's 73-100, B's 63-72, C's 53-62, D's 43-52, F's below 43.

The worst scores were on problem 1b), which is exercise 6.9.12a. It's very similar to an assigned HW problem (6.9.11). The scores were also low on problem 4a, which was admittedly a bit tricky (though 4b was not), and on problem 3, fairly straightforward integrals, mostly from the Ch 8 exercises. Practice these (especially 3 and 1b) before the next exam! The True-False went unusually well (but these were very similar to practice ones on my web site).

1a)  $\lim_{x \rightarrow +\infty} (1 + \frac{1}{x})^x = e$ . See Thm 6.9.8.

1b) Set  $x = 3y$  in the 1b limit to get  $e^3$ , from the 1a limit. There are at least 2 other possible methods, based on the various options in 1a.

2) Let  $y$  be the height (from the bottom of the tank). Then  $r = y/3$ , and  $\text{dist} = 12 - y$ .  $W = \int_0^8 62 \pi (y/3)^2 (12 - y) \, dy = \frac{62}{9} \cdot \pi \cdot 2 \cdot 8^3$ . It is *not* OK to leave  $|_0^8$  in the answer.

3a)  $\frac{x^2}{2} \ln(x) - \frac{x^2}{4} + C$  Use IBP.

3b)  $\frac{-\cos^3(x)}{3} + \frac{\cos^5(x)}{5} + C$  (set  $u = \cos(x)$ ).

3c)  $\frac{5^x}{\ln(5)} + C$  (memory, or set  $5^x = e^{x \ln 5}$ ,  $u = x \ln 5$ , etc)

3d)  $x \sin^{-1}(x) + \sqrt{1 - x^2} + C$  Use IBP.

3e)  $\tan^{-1} |_{-1}^1 = \pi/2$  (memory, or set  $x = \tan u$ , etc).

4a)  $L = \int_1^8 \sqrt{1 + \frac{4}{9}x^{-2/3}} \, dx = \int_1^8 x^{-1/3} \sqrt{x^{2/3} + \frac{4}{9}} \, dx = [x^{2/3} + \frac{4}{9}]^{3/2} |_1^8 = ([4 + 4/9]^{3/2} - [1 + 4/9]^{3/2})$  (set  $u = x^{2/3} + 4/9$ ). One clever student changed the problem to  $x = y^{3/2}$ ,  $0 \leq y \leq 2$ . This leads to easier algebra (but he didn't finish it correctly), similar to ex 7.4.3. This was problem 7.4.5.

4b) With simple algebra (such as  $\sqrt{a}\sqrt{b} = \sqrt{ab}$ ) the integral becomes  $2\pi \int_{-1}^1 \sqrt{4} dx = 8\pi$ . But be careful! After a single mistake, the problem may become impossible to finish. This was 7.5.3.

5) You should state which method you are using (commenting on your work is simply a good habit, and it often helps with partial credit). You can use shells (dx) or washers (dy). With shells,  $V = \int_0^4 2\pi x\sqrt{x} dx = 128\pi/5$ .

6) TFFFT

7a) The volume of a washer centered on the  $x$  axis is  $V = Ah = \pi(R^2 - r^2)\Delta x$ , where  $R = f(x)$  and  $r = g(x)$  (this, and the next step, should be illustrated with pictures, which I can't easily include here). So, a solid which can be split into washers has volume  $V \approx \sum_{k=1}^n \pi[(f(x_k^*))^2 - (g(x_k^*))^2]\Delta x_k$ . After taking a limit, as  $n \rightarrow \infty$ , the formula becomes exact, and the Riemann sum becomes an integral,

$$V = \int_a^b \pi[(f(x))^2 - (g(x))^2] dx$$

7b) Begin with  $\ln(ab) = \int_1^{ab} \frac{1}{t} dt$ , etc, as in class. Mention the definition 3 times, and the abc theorem once, and show the steps of the u-substitution. This is pretty straightforward, and the results were pretty good on this one. Nobody chose 7c.

Bonus: Quite a few people got this one, though I thought it might be tough. Start with  $\int (1 - \sin^2(x)) \cos(x) dx$  and set  $u = \sin(x)$ . Get  $\sin(x) - \sin^3(x)/3 + C$ .