Exam III and Key — Nov 8, 2007

MAC 2312 — S Hudson

You may use these formulae whenever needed:

$$\int \sec^n(x)\ dx = \frac{\sec^{n-2}(x)\tan(x)}{n-1} - \int \sec^{n-2}(x)\ dx$$

$$\int u^n \ln u\ du = \frac{u^{n+1}}{(n+1)^2}[(n+1)\ln u - 1] + C$$

$$|E_M| \le \frac{(b-a)^3 K_2}{24n^2} \quad \text{and} \quad |E_T| \le \frac{(b-a)^3 K_2}{12n^2}$$

1) (15pts) For each series, answer either Converges (C) or Diverges (D), show your work, and state which "test" you are using.

a) $\sum_{k=1}^{\infty} \frac{1}{k^{1/4}}$

b) $\sum_{k=1}^{\infty} \cos(\pi k)$

c) $\sum_{k=1}^{\infty} \sin(\pi k)$

2) (10pts) Answer True or False:

The harmonic series converges.

Every decreasing sequence that is bounded below converges.

The Ratio Test is conclusive for the p-series when $p = 2$.

$\int_{-\infty}^{\infty} \sin(x)\ dx = 0$

$a_n = 50\ln(n) - n$ $(n \ge 1)$ is eventually monotonic.

3) (45pts) Compute the integrals (or write "diverges"). Show all your work.

a) $\int \frac{1}{x^2-4x+5}\ dx$

b) $\int_{-1}^{\infty} \frac{x}{1+x^2}\ dx$

c) $\int \sec^3 x\ \tan^2 x\ dx$

d) $\int_{-1}^{+1} \frac{dx}{x^{2/3}}$

e) $\int \frac{5x-5}{3x^2-8x-3}\ dx$

f) $\int \frac{x^2}{x^2+4}\ dx$

g) $\int \frac{\ln(x)}{\sqrt{x^3}}\ dx$

h) $\int \ln(2x+1)\ dx$

i) $\int \sin^3 x\ \ \cos^2 x\ dx$

4) [5pts] Use a_{n+1}/a_n to show that $\{ne^{-n}\}_{n=1}^{\infty}$ is strictly monotonic.

5) [5pts] Find the sum of the series (or write "Diverges"); $\sum_{k=0}^{\infty} \frac{4^{k+2}}{7^{k-1}}$

6) [5pts each] a) Use the Trapezoid Rule with $n = 4$ to estimate $\int_0^2 \frac{1}{2x+1}\ dx$. You may leave +'s and /'s in your answer. But for a little extra credit, simplify and check your answer (using the FTC). Hints/Approxns: $1/3 = .33$, $1/6 = .17$, $\ln(5) = 1.61$.

6b) Find the smallest value of n so that the error estimate (see pg 1) for this integral ensures an absolute error less than 10^{-6}. Hints: You can leave a $\sqrt{}$ in your answer. I noticed that f'' is decreasing, and when I calculated K_2, I got an even integer.

7) (10pts) Choose ONE proof, explain thoroughly:

a) Calculate $\int \sec(x)\ dx$, explaining each step.

b) State and prove the Divergence Test.

c) State and prove the Comparison Test.

Remarks and Answers: The average was about 60/100. The unofficial scale is A's 74-100, B's 64-73, C's 54-63, D's 44-53. The grades were best on problems 1 and 2; worst on problem 4. Also, review integration methods (problem 2) before the final. The semester average without HW is now about 62/100.

1a) D; it's a p-series with $p = 1/4 < 1$.

1b) D, by Div.T. ($\lim a_k = \lim(-1)^k$ d.n.e.)

1c) C (every $a_k = 0$).

2) FTFFT The Ratio Test is inconclusive for every p-series, because it always produces $\rho = 1$ for them.

3a) $\tan^{-1}(x-2) + C$, by completing the square.

3b) Diverges. Your work should include a limit and the substitution $u = 1 + x^2$. Remember that "Diverges" can only be correct for a definite integral like this one, not for problems 1a, 1c, etc.

3c) $\int \sec^5 x - \sec^3 x = (\sec^3 x \tan x)/4 - \sec x \tan x + 2\ln|\sec x + \tan x| + C$, by using the reduction formula on page 1.

3d) 6. It's improper, so you must use limits for full credit.

3e) P.Fracs: $\int \frac{2}{3x+1} + \frac{1}{x-3} = (2/3)\ln|3x+1| + \ln|x-3| + C$

3f) $\int 1 - \frac{4}{x^2+4}\, dx = x - 2\tan^{-1}(x/2) + C$

3g) Use the mini-table on page 1 and get $4x^{-1/2}[(-1/2)\ln(x) - 1] + C$. It is also OK to use IBP. Either way, be careful! There were *lots* of silly mistakes on this one.

3h) Use IBP with $g(x) = \ln(2x+1)$ and $f'(x) = 1$. If you do the obvious thing, and set $f(x) = x$, you should get $x\ln(2x+1) - \int \frac{2x}{2x+1}\, dx = x\ln(2x+1) - x + (1/2)\ln(2x+1) + C$ (divide, as usual with partial fractions). A less obvious thing, but a bit easier, is to set $f(x) = x + 1/2$ (try it!).

A few people used the same mini-table formula that they used in 3g (but with $n = 0$ and $u = 2x+1$). I didn't expect that, but it is OK.

3i) $\cos^5(x)/5 - \cos^3(x)/3 + C$

4) This is exercise 10.2.9.

Step 1: Calculate, $a_{n+1}/a_n = \frac{(n+1)e^{-n-1}}{ne^{-n}} = (1+\frac{1}{n})e^{-1}$. Many people made algebra errors here, such as writing e^{-n+1}. Or, they took a limit.

Step 2: We'll show this is always less than 1. Since $n \geq 1$, we know $1/n \leq 1$, so $a_{n+1}/a_n \leq (1+1)/e < 1$.

Step 3: (state your conclusion clearly, with reasoning) Since $a_{n+1}/a_n < 1$ for all n, we know $a_{n+1} < a_n$ and the sequence is strictly decreasing.

5) $\frac{4^{k+2}}{7^{k-1}} = \frac{4^2}{7^{-1}} \cdot (\frac{4}{7})^k$ so it is geometric, with $a = \frac{4^2}{7^{-1}} = 112$ and $r = \frac{4}{7} < 1$. So, $S = a/(1-r) = 784/3$.

6a) $T_4 \approx 3.36/4 = .84$

6b) To get K_2, notice that $|f''(x)| = 8(2x+1)^{-3}$, which *decreases* for $x \geq 0$. So the max is $K_2 = f''(0) = 8$. Plugging this in, and setting $|E_T| \leq 10^{-6}$, we get $n \geq \sqrt{64 \cdot 10^6/12} = 4000/\sqrt{3}$. Actually, n must be an integer, and this number must be rounded up. But I didn't mark off for that. Your combined grade for 6a+6b is in the left margin, out of ten points.

7) See the text or lecture notes.