MAC 2312 Dec 11, 2007
Final Exam and Key Prof. S. Hudson

1) (10pts) For each series, answer either Converges absolutely (CA) or Converges conditionally (CC) or Diverges (D). State which "test" you are using, and show your work.

a) $\sum_{k=1}^{\infty} \cos(k)/k^2$

b) $\sum_{k=1}^{\infty} \frac{1}{k \ln k}$

2) (9pts) A cylindrical water tank of radius 5 ft and height 9 ft is two-thirds filled with water [density = 62 lbs/ft^3]. Find the work required to pump all the water over the upper rim.

3) [5 pts each] Compute (or explain why no answer exists):

$\int_1^e \ln(x)\ dx$

$\int \cos^3 x \sin^2 x\ dx$

$\int 2^x\ dx$

$\int_0^{2\pi} |\sin(x)|\ dx =$

$\int_{-\infty}^{+\infty} \frac{1}{x^2+2x+5}\ dx$

4) (5pts) Write down, but do not evaluate, an integral for the area of region enclosed by the rose $r = \sin(3\theta)$.

5) (10pts) Let $x = \sqrt{t}$ and $y = 2t + 4$.

a) Find dy/dx when $t = 1$, without eliminating the parameter.

b) For the same curve, find d^2y/dx^2 when $t = 1$, without eliminating the parameter.

6) (15pts) a) Find the McLaurin Series for the function $f(x) = \sin(2x)$.

b) State LaGrange's estimate for $R_n(x)$. Then estimate $R_5(1)$.

c) Show that the series you found in part a) converges to $\sin(2x)$ for all x.

7) [16 pts] Answer True or False:

The area inside $r = \sin 4\theta$ is equal to the area inside $r = \cos 4\theta$.

If $u_k \to 0$ then $\sum_1^{\infty} u_k$ converges.

The average value of $x(x-4)$ on $[0,4]$ is zero.

If $0 < a_n < 1$ for all n, then $\{a_n\}$ converges.

The Trapezoid Rule is exact if f is a polynomial of degree 2.

There are exactly 8 points on the graph of $r = \cos(2\theta)$ where the tangent line is horizontal.

A bounded decreasing sequence must converge.

The interval of convergence of a McLaurin series must be centered at 0.

8) [10 pts] Choose ONE:

a) State and prove the integration formula for area in polar coordinates. Draw a picture showing α, β, etc., and *use a* $\sum$ *and a limit* in your explanation.

b) State and prove the Comparison Test.

Remarks and Answers: The average score was about 61. Roughly, A's started at 75, B's at 65, and so on. You can see me early in Spring term to look at your final. The best scores were on problems 2, 3 (but not 3e) and 7 (TF). The worst scores were on problems 3e, 5b and 6bc; and 8 was a bit low too.

1a) CA; Put on abs vals, and compare it with a p-series (p=2).

1b) D; by the integral test (substitute $u = \ln(x)$).

2) $W = \int_0^6 62\pi 5^2(9-y)\,dy = 62\pi \cdot 25 \cdot 36$ ft lbs

3a) $x\ln x - x|_1^e = 1$

3b) set $u = \sin x$, get $(\sin x)^3/3 - (\sin x)^5/5 + C$

3c) $2^x/\ln 2 + C$. I accepted "by memory" for showing work on this one.

3d) $\int_0^\pi \sin x\,dx - \int_\pi^{2\pi} \sin x\,dx = 2 + 2 = 4$, or use symmetry (for smaller chance of silly errors); $2\int_0^\pi \sin x\,dx = 4$.

3e) $\pi/2$ - complete the square, handle improper, set $x + 1 = 2\tan\theta$, etc.

4) Using symmetry, $A = 3/2\int_0^{\pi/3} \sin^2\theta\,d\theta$ (other answers are possible).

5a) $4t^{1/2}|_{t=1} = 4$

5b) 4 (see Ex 4 page 733 and the assigned HW problem, 11.2.5).

6a) $2x - 8x^3/3! + 32x^5/5! + \ldots$ There was a typo in this answer, from 2007 until 2011, when some students studying for their final finally caught it. Note to self; apparently, after taking the final, my students are not very interested in these answer keys! (SH: 8/20/11).

6b) $|x - x_0|^{n+1} M/(n+1)!$

The center is $x_0 = 0$. The Chain Rule produces another 2 every time you take a derivative, so $M = 2^{n+1}$ [not $M = 1$ this time!], and $|R_5(1)| \leq 2^6/6!$

6c) Show that $\forall x, |R_n(x)| \leq |2x|^{n+1}/(n+1)! \to 0$. I stated during the exam that it was *not* OK to quote facts about the series for $\sin(x)$ here; I wanted to test you on using R_n. Otherwise, that idea would also work. (see 10.9 Ex 1).

7) TFFF FFTT; Note on 4th one: $\{a_n\} = .2, .3, .2, .3, \ldots$ diverges. On the 5th: The Trapezoid rule is exact for linear functions, and Simpson's is exact for quadratics. These comments should not be memorized; think about them instead.

8) See text/lectures.