Exam I, 100 minutes, (Key) Jan 30, 2003
MAC 2312 S Hudson

The average score was about 64/100 (omitting scores below 45). The weak spots on the exam were 1e and 1h. And maybe 1a, 1d, 1j and 5.

Most of the problems in part 1) were exercises from the textbook. We will do more calculations like these, but a little harder, in Chapter 7.

a) $\frac{d}{dx}$ [$\int_0^x \sec^2(t)\ dt$]

Use the FTC ! It's just $\sec^2(x)$. [The long way is $\frac{d}{dx}$ $[\tan(x) - \tan(0)] = \sec^2(x)$].

b) $\int_0^4 \sqrt{x} + e^x\ dx$

$2x^{3/2}/3 + e^x|_0^4 = 16/3 + e^4 - 1$

c) $\int_1^2 \frac{1}{t}\ dt$

$\ln 2 - \ln 1 = \ln 2$

d) $\int_0^{\pi/2} \sin^2 x\ \cos x\ dx$

Set $u = \sin x$ and get $\int_0^1 u^2\ du = 1/3$

e) $\int_0^3 |x-2|\ dx =$

$\int_0^2 -(x-2)\ dx + \int_2^3 x - 2\ dx = 2 + 1/2 = 5/2.$

f) Find the average value of $f(x) = x^3$ on the interval [0,5].

$\frac{1}{5}\ x^4/4\ |_0^5 = 625/20 = 125/4.$

g) $\int_0^4 \frac{dx}{\sqrt{2x+1}} =$

Set $u = 2x + 1$, and get $\frac{1}{2}u^{1/2}/\frac{1}{2}\ |_1^9 = 3 - 1 = 2$. NOTE: Several people wrote $u^{1/2}|_0^4 = 2 - 0$, which is wrong. By coincidence, it leads to the same answer.

h) $\int_0^\pi \sin^2 3x\ dx$

$= \int_0^\pi (1 - \cos(6x))/2\ dx = \pi/2$. If you got 0 (or a negative answer), you should have done a "sanity check".

i) Find the area between the parabolas, $y = 2x - x^2$ and $y = 2x^2 - 4x$.

Answer: 4. [Suggestion: Draw a graph.] From $2x - x^2 = 2x^2 - 4x$, we get $x = 0$, $x = 2$, the limits of integration. Then $\int_0^2\ 2x - x^2 - (2x^2 - 4x)\ dx = 4$. Common mistake: forgetting to put parentheses around $2x^2 - 4x$! Which leads to an impossibly negative answer.

j) Use left-hand endpoints to find $\underline{A}_2$ for $f(x) = \sin(x)$ on $[0, \pi/2]$.

Answer: [Suggestion: Draw a graph]. Then get $\Delta x = \pi/4$ and use $x_0 = 0$ and $x_1 = \pi/4$ (ignore $x_2 = \pi/2$). So, $(\pi/4)[\sin(0) + \sin(\pi/4)] = \sqrt{2}\pi/8$.

2) (10pts) Rosanne drops a ball from a height of 400 ft. The force of gravity is 32ft/s/s, which is the acceleration on the ball. So, the velocity is $-32t$ (or, you might use $32t$).

a) Find a formula for the height of the ball $h(t)$ after t seconds. Be sure to show all your work and reasoning, and check that $h(0) = 400$.

Answer: $h(t) = \int v(t) = -16t^2 + C = -16t^2 + 400$. [Or, you can use "distance travelled" $= \int 32t = 16t^2$ and subtract this from the initial height, 400]. If you answer with a formula such as " $-16t^2 + v_0 t + C$ ", you must state clearly that $v_0 = 0$ and $C = 400$.

b) How many seconds until the ball hits the ground? (This occurs when h=0, of course).

Answer: Set $0 = h = -16t^2 + 400$, and get $t = 5$.

3) (10pts) a) Let $f(x) = x^2$ on [0,2]. Use the well-known formula $n(n+1)(2n+1)/6$ to calculate and simplify the Riemann Sum below. Answer in terms of n

$$\sum_{i=1}^{n} f(x_i)\Delta x$$

Answer: $\Delta x = 2/n$, so $x_i = 2i/n$, and get

$$\sum_{i=1}^{n} (2i/n)^2 2/n = (8/n^3)\sum_{i=1}^{n} i^2 = 8n(n+1)(2n+1)/6n^3$$

b) Find the exact area under this curve by taking a limit as $n \to \infty$. [Do NOT use the FTC, except maybe to check your answer].

Answer: 8/3

4) (15 pts) Answer True or False in the margin. You don't have to explain. You can set $n = 10$ in the last three if you like.

There is a point c in [0,1] such that $\sin(c) = \int_0^1 \sin(x)\, dx$.

$f(x) = 1/x$ is integrable on the interval [-1,1].

If f is linear, then Simpson's rule will produce an exact answer for $\int_a^b f(x)\, dx$.

The midpoint rule produces an exact answer for $\int_0^2 3x^2\, dx$.

The midpoint rule produces a smaller estimate for $\int_1^4 5 - x^2\, dx$ than the Trapezoid rule.

Answer: TFTFF

5) (15pts) Choose ONE proof. Explain thoroughly. You can continue on the back.

A) State and prove the FTC, Part One, about computing the derivative of a definite integral. [You can use other definitions and theorems in your proof - but point out when you are doing so]

Answer: The proof is in the book. Common mistakes were:

a) never stating the theorem (including the hypothesis that f is continuous, the formula for $F(x)$ etc).

b) weak explanations. Good explanations should include the definition of derivative, the interval union theorem, the definition of average value, and the associated theorem, and that f is continuous.

c) Calculation or notational errors

Next time, be sure to practice each proof a few times and then compare with the text or the lecture notes.

B) State and prove the formula for $1 + 2 + \ldots + n$.

Answer: See equation 7 on page 321. The idea of the proof is given in exercise 5.3.43 (and was done in class).