MAD 4203 June 19, 2008
Final Exam and Key Prof. S. Hudson

1) How many ways can you tile a 2x5 board with dominoes? Show work.

2) [re-typed] Let h_n be the number of sequences of length n formed from 0, 1 and 2, such that no 2 is followed eventually by a 1. For example, 10202 is OK but not 10201. Find a recurrence relation for the h_n and use it to compute h_3. Hint: Look at three cases, depending on how the sequence starts: 0,1 or 2. How is the rest completed? Suggestion: Check your answer for h_3 by listing - if you have time.

3) How many ways can you place 5 non-attacking rooks on a 8x8 chessboard such that neither the first row nor the first column is empty?

4) Compute the derangement number D_6. Use any identity from the book, but no calculator.

5) Find the generating function for h_n, where $h_n = 8h_{n-1} - 16h_{n-2}$ and $h_0 = -1$ and $h_1 = 0$. Leave your answer in rational form [like $g(x) = (x - 1)/(x^2 - 9)$]. You don't have to use partial fractions or solve for h_n.

6) Find a formula for $\sum_{k=0}^{n} k^4$ using differences. Leave your answer as a short sum of binomial coefficients. The first few k^4 numbers are 1, 16, 81, 256, 625 (etc). The first differences are 15, 65, 175, 369 (etc).

7) Let $h_n^{(k)}$ be the number of regions in R^k formed by n hyperplanes of dimension $k - 1$ in general position [as in Ch 8.4]. For example, $h_3^{(2)} = 7$. Find $h_5^{(3)}$ (you don't have to prove your answer, but show your work).

8) Apply the Matching Algorithm to M shown below. Either find a bigger matching M^2, or find a cover S, so that $|S| = |M|$.

9) Apply the deferred acceptance algorithm (aka Gale-Shapley algorithm) to find a stable marriage for the ranking matrix below. Imagine that the women are A, B, C, D and the men are a, b, c, d and that the women choose the men (so your answer will be women-optimal).

$$\begin{pmatrix} 1,2 & 2,1 & 3,2 & 4,1 \\ 2,4 & 1,2 & 3,1 & 4,2 \\ 2,1 & 3,3 & 4,3 & 1,4 \\ 1,3 & 4,4 & 3,4 & 2,3 \end{pmatrix}$$

10) Choose ONE.

a) State and prove thm 9.2.1 (about max matchings and paths)

b) State and prove thm 9.3.2 (about SDR's and the MC).

Bonus: [about 5 points] How many ordered pairs (A, B) of subsets of $S = \{1, 2, 3 \ldots n\}$ $(n \geq 2)$ have the property that $|A \cap B| = 2$?

Remarks: This was a 75-minute final, but nobody needed our classroom, so I let it go over. The average was about 69/100, with two grades over 95. The scores were almost perfect on 1 (counting tilings), 8 and 9 (applying Ch 9 matching algorithms). Rather low on 2, 3 and 7. Average scores: 10, 5.9, 4.7, 8.3, 7.2, 6.9, 5.6, 9.6, 9.5, 7.7, 0.45/5, 69/100.

My first version of the exam had a different Problem 2: Find a maximal clutter in $S = \{1, 2, 3, 4\}$ [your answer should be a list of subsets of S].

1) 8 (use recursion and get the 5th Fibonnacci number) see also Ch.1 HW.

2) $h_n = 2^{n-1} + 2h_{n-1}$, and $h_3 = 20$. As I recall, nobody checked h_3 (or even h_2) by listing. Several silly mistakes could've been caught this way.

3) Direct Method: With one rook in position (1,1) there are $C(7, 4)^2$ ways to choose the remaining rows and columns for the rooks, and 4! ways to place them. Otherwise there are 7^2 ways to place the first two rooks and $C(6, 3)^2$ times 3! ways to finish. Total $4!C(7, 4)^2 + 49x3!C(6, 3)^2 = 294110 + 1176110 = 1470110$.

IE Method: There are $5!C(8, 5)^2$ ways to place 5 rooks, ignoring the 1st row / column rule. Then, subtract $2C(8, 5)C(7, 5)5!$ and add back in $5!C(7, 5)^2$. Also get $376320 - 282240 + 52920 = 1470110$ (but simplification was not required).

4) 265. Most people got this, by computing $D_1 = 0$ and $D_2 = 1$ (by listing), then $D_3 = (3 - 1)(D_1 + D_2) = 2$ (by recursion) etc, up to D_6.

A few people worked backwards in this problem and/or in Problem 2. For example, $D_6 = 5(D_5 + D_4) = 5(4(D_4 + D_3) + 3(D_3 + D_2)) =$ etc. But this is very awkward - not the way recursion is intended to be used - and very likely to lead to errors.

5) $g(x) = (-1 + 8x)/(1 - 8x + 16x^2)$

6) I mentioned that you can insert the term $k^4 = 0$ if you like, and this probably makes the problem a little easier. Then, the first diagonal is 0, 1, 14, 36, 24, 0, 0 etc. We get $0 \cdot C(n + 1, 1) + 1 \cdot C(n + 1, 2) + 14 \cdot C(n + 1, 3) + 36 \cdot C(n + 1, 4) + 24 \cdot C(n + 1, 5)$

Without inserting 0, the diagonal is 1, 15, 50, 60, 24, 0, 0, etc, and the answer takes the form $1 \cdot C(n, 1) + 15 \cdot C(n, 2) + 50 \cdot C(n, 3) + 60 \cdot C(n, 4) + 24 \cdot C(n, 5)$

7) 26 (see ch 8.4)

8) The M-a path is $x_1y_2x_3y_5x_4y_1$. Remove y_2x_3 and y_5x_4 from M and add in the other three edges of the path to get $M^2 = x_1y_2$, x_3y_5, x_4y_1, x_2y_4 and x_5y_3.

9) Finish with pairs Bc, Ab, Ca, Dd.

10) See text. Most people chose b).

Bonus - very few people tried this, but it is similar to two previous quiz problems: $C(n, 2)3^{n-2}$.