Key to Final Exam, Linear Algebra, Fall 2002

solution to problem 1 corrected, 12/12/03

1a) Compute X^{-1} using GE (or the adjoint) and check that

$$XDX^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = A$$

1b) You do *not* have to compute A^{-1}. It's e-things are related to A's. The eigenvalues are 1/2 and -1 (see D) and the eigenvectors are the same as for A, namely $\mathbf{x_1} = [1,1]^T$ and $\mathbf{x_2} = [1,-2]^T$ (see X). See HW 6.1.4.

1c)

$$e^A = Xe^DX^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} e^2 & 0 \\ 0 & e^{-1} \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} = \text{etc}$$

1d) (see 6.3.4) There is more than one right answer, but the simplest is:

$$B = A^{1/2} = XD^{1/2}X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$$

2) (see 6.4.5c) Expanding along the bottom row,

$$p(\lambda) = (2-\lambda)[(2-\lambda)^2 - 1] = (2-\lambda)[3 - 4\lambda + \lambda^2] = (2-\lambda)(1-\lambda)(3-\lambda)$$

So the eigenvalues are 1,2 and 3 (you can use any order). Now find $N(A-\lambda I)$ (using GE or guessing) for each of these and get $x_1 = [i,-1,0]^T$ and $x_2 = [0,0,1]^T$ and $x_3 = [i,1,0]^T$ (scalar multiples of these are also OK). These are already orthogonal, since A is Hermitian, but you must divide by their lengths before putting them into P. Again, there are other correct answers, but I get

$$A = PDP^H = \begin{bmatrix} i/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ i/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -i/\sqrt{2} & 0 & -i/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

3) This is 5.3.3a. Solve $A^T Ax = A^T b$, which often produces a unique answer, but not this time.

$$A^T A = \begin{bmatrix} 6 & 12 \\ 12 & 24 \end{bmatrix} \quad A^T b = \begin{bmatrix} 6 \\ 12 \end{bmatrix} \quad \text{so} \quad \mathbf{x} = \begin{bmatrix} 1 - 2\alpha \\ \alpha \end{bmatrix}$$

4) FTTFF TFTTT

5) These are in the textbook.

$$6) \text{ adj } A = \begin{bmatrix} 6 & -9 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 2 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 1 & -3/2 & 1/3 \\ 0 & 1/2 & -2/3 \\ 0 & 0 & 1/3 \end{bmatrix} \quad \det A^{-1} = 1/6$$

7) rank = 2, nullity = 5 - 2 = 3.

8) [Simple exercise, dismal results] A square matrix Q is orthogonal iff $Q^T Q = I$. So, assume this, and the same for another matrix S. Then $(QS)^T QS = S^T Q^T QS = S^T IS = I$, so QS is orthogonal too.

You really need to know the simple defining formulas for "symmetric", "orthogonal", "normal" etc.

Bonus (see final from summer 2000, and the web page "my favorite matrices"): Let L be a 120^o rotation. Since 3x120 = a full rotation, $L^3 = id$. Set $A =$ the matrix rep of L (and find it).