Partial Key to the Final Exam, Linear Algebra, Spring 2003

(The questions will be posted separately)

1) Proofs about "orthogonal" or "symmetric" etc are usually easy, because these words can be defined by equations. Which means the proof is mainly a calculation. In this problem, *do not* talk about perpendicular vectors etc. Instead:

Answer: If A and B are orthogonal, we know that $A^T A = I$ and $B^T B = I$. We must check this for AB. From matrix algebra and the assumptions, we get $(AB)^T \ AB = B^T A^T AB = B^T IB = I$, as desired.

2) Do the usual 6.1 calculations. Get $\lambda = 1$ or 3 and put the e-vecs into a matrix X. Then normalize the columns to get:

$$U = 2^{-1/2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

3) We must use *the definition* of subspace here. 1) [closure of scalar-mult] Assume $\mathbf{v} \in U \cap V$ (so, it is in U and in V) and that $\alpha \in R$ is a scalar. Since $\mathbf{v} \in U$ and U is a subspace, we know $\alpha\mathbf{v} \in U$. Likewise, we can show $\alpha\mathbf{v} \in V$, so $\alpha\mathbf{v} \in U \cap V$.

I leave part 2) to you [closure under addition].

4) I expected you to do the usual - to compute the columns one-at-a-time. Another simple method is to set $AB = C$, where the columns of B are the input vectors and the columns of C are the output. Then

$$A = CB^{-1} = \begin{bmatrix} 2 & 6 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

5) FTTTF TTFTF

Due to technical problems I must end the key here. I'll finish it later if I hear from enough interested people.

1a) Compute X^{-1} using GE (or the adjoint) and check (*and do it honestly*) that

$$XDX^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} = A$$

1b) You do *not* have to compute A^{-1}. It's e-things are related to those of A. The eigenvalues are 1/2 and -1 (see D) and the eigenvectors are the same as for A, namely $\mathbf{x_1} = [1,1]^T$ and $\mathbf{x_2} = [1,-2]^T$ (see X). See exercise 6.1.4.

1c)

$$e^A = Xe^DX^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} e^2 & 0 \\ 0 & e^{-1} \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} = \text{etc}$$

1d) (see 6.3.4) There is more than one right answer, but the simplest is:

$$B = A^{1/2} = XD^{1/2}X^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix}$$

2) (see 6.4.5c) Expanding along the bottom row,

$$p(\lambda) = (2-\lambda)[(2-\lambda)^2 - 1] = (2-\lambda)[3 - 4\lambda + \lambda^2] = (2-\lambda)(1-\lambda)(3-\lambda)$$

So the eigenvalues are 1,2 and 3 (you can use any order). Now find $N(A-\lambda I)$ (using GE or guessing) for each of these and get $x_1 = [i,-1,0]^T$ and $x_2 = [0,0,1]^T$ and $x_3 = [i,1,0]^T$ (scalar multiples of these are also OK). These are already orthogonal, since A is Hermitian, but you must divide by their lengths before putting them into P. Again, there are other correct answers, but I get

$$A = PDP^H = \begin{bmatrix} i/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ i/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -i/\sqrt{2} & 0 & -i/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

3) This is 5.3.3a. Solve $A^TAx = A^Tb$, which often produces a unique answer, but not this time.

$$A^TA = \begin{bmatrix} 6 & 12 \\ 12 & 24 \end{bmatrix} \quad A^Tb = \begin{bmatrix} 6 \\ 12 \end{bmatrix} \quad \text{so } \mathbf{x} = \begin{bmatrix} 1-2\alpha \\ \alpha \end{bmatrix}$$

4) FTTFF TFTTT

5) These are in the textbook.

$$6)\ \text{adj } A = \begin{bmatrix} 6 & -9 & 2 \\ 0 & 3 & -4 \\ 0 & 0 & 2 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 1 & -3/2 & 1/3 \\ 0 & 1/2 & -2/3 \\ 0 & 0 & 1/3 \end{bmatrix} \quad \det A^{-1} = 1/6$$

7) rank = 2, nullity = 5 - 2 = 3.

8) [Simple exercise, dismal results] A square matrix Q is orthogonal iff $Q^T Q = I$. So, assume this, and the same for another matrix S. Then $(QS)^T QS = S^T Q^T QS = S^T IS = I$, so QS is orthogonal too.

You really need to know the simple defining formulas for "symmetric", "orthogonal", "normal" etc.

Bonus (see final from summer 2000, and the web page "my favorite matrices"): Let L be a 120^o rotation. Since 3x120 = a full rotation, $L^3 = id$. Set A = the matrix rep of L (and find it).