MAS 3105 April 19, 2004
Final Exam and Key Prof. S. Hudson

1) (10pts) Suppose that A is a square singular matrix. What can you say about the product A adj A ?

2) (10 pts) a) Find the least squares solution to $A\mathbf{x} = \mathbf{b}$, using this QR factorization:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & -4 & 2 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{where } \mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

2b) Let $S = R(A)$. Find $\text{proj}_S \mathbf{b}$.

3) (15pts) a) Given that A is row equivalent to U, find a basis for $N(A)$. Check !

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix} \quad U = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3b) Find a basis of the row space of A.

3c) Find a basis of the column space of A.

4) (10pts) Find the standard matrix representation for each of the following linear transformations,

a) L rotates each vector $\mathbf{x}$ in R^2 by 45° in the clockwise direction.

b) L reflects each vector about the line $x_1 = x_2$, and then projects it onto the x_1 axis.

5) (10pts) Show that the product of two unitary matrices is unitary. [Hint - instead of the definition of unitary, use a formula with U^H in it.]

6) (10 pts) Find the projection of x onto $\sin(2x)$ using the inner product,

$$< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\ dx$$

7) (20 points) Answer True or False. Assume the matrices are 3x3 (if it matters).

Every Hermitian matrix is diagonalizable.

If V has a spanning set of 5 vectors then dim $(V) \leq 5$.

The product of two Hermitian matrices is Hermitian.

The sum of two Hermitian matrices is Hermitian.

Every unitary matrix is normal.

If A and B have the same eigenvalues 2, 4 and 8, then A is similar to B.

If 0 is an eigenvalue of A, then the nullity of A is positive.

For every A, the matrices AA^T and A^TA are both symmetric.

If r is an eigenvalue of A, then r^2 is an eigenvalue of A^2.

The vector space R^{3x3} of all 3x3 matrices has dimension 9.

8) (5pts) Which MATLAB command(s) could you use to quickly create a 6x6 matrix with rank = 3? Use "rand" (and don't worry about any unlikely results).

9) (10pts) Choose ONE:

A) Prove that any two similar square matrices A and B must have the same eigenvalues.

B) State and prove the Spectral Theorem (6.4.4).

Bonus (5 pts): Give examples of 3x3 matrices A and B such that rank $AB \neq$ rank BA (or prove this is not possible).

Answers: The average was about 56/100. The lowest scores were on 2, 5 and 6.

1) A adj $A = (\det A)I = O$ (the zero matrix).

2a) Set $R\hat{x} = Q^T b = [-1, -1, 2]^T$ and get $\hat{x} = [-2/5, 0, 1]^T$. See Ex 4, pg 296.
2b) Set $p = A\hat{x} = [2/5, 1/5, 3/5, -8/5]^T$.

3a) The free variables are $x_4 = \alpha$ and $x_3 = \beta$. Solving for the others, the nullspace is the set of $\alpha[-7, -3, 0, 1, 0]^T + \beta[-3, -1, 1, 0, 0]^T$ and the two vectors in that formula form a basis of $N(A)$.

3b) The 3 nonzero rows of U.

3c) $\{\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_5}\}$. This is Ex 3, page 179.

4) Let $\sin(\pi/4) = \cos(\pi/4) = .7$.

$$A = \begin{bmatrix} .7 & .7 \\ -.7 & .7 \end{bmatrix} \quad \text{and} \quad = A \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

5) Assume U and V are unitary so that $U^H U = I$ and $V^H V = I$. [Obviously, you need to *know* this formula !] To check this for the product, $(UV)^H UV = V^H U^H UV = V^H IV = I$.

6) $proj = -\sin(2x)$. Since $\sin(2x)$ has norm 1, the projection is $\alpha \sin(2x)$ where

$$\begin{aligned} \alpha = < x, \sin(2x) > =& \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(2x)\ dx \\ =& \frac{1}{\pi} [-x \cos(2x)/2|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \cos(2x)\ dx \\ =& -1/2 - 1/2 = -1 \end{aligned}$$

7) TTFTT TTTTT

8) rand(6,3)*rand(3,6)

9) See text

Bonus: Almost everyone thought this was impossible! If A is nonsingular, then rank AB = rank B = rank BA. So, we need two singular matrices. Here is a simple plan: Make $AB = O \neq BA$ [see Ch1 HW]. To be specific, let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$