MAS 3105 June 22, 2005
Final Exam Prof. S. Hudson

Name

Show all your work and reasoning for maximum credit. If you continue your work on another page, be sure to leave a note. Do not use a calculator, book, or any personal paper. You may ask about any ambiguous questions or for extra paper. If you use extra paper, hand it in with your exam.

1) [20pts] Set $B = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\} = \{(1,0,0)^T, (1,0,1)^T, (0,1,1)^T, (0,0,1)^T\}$. So, B is a set of 4 column vectors in R^3. Answer, and explain briefly:

a) Is B a spanning set of R^3?

b) Is B linearly independent?

c) Let $C = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_4}\}$. Is C a basis of R^3?

d) Let $D = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$. Show that D is a basis of R^3.

e) Find the transition matrix from D to the standard basis of R^3.

2) [15 points] Choose ONE of these to prove (you can use the back).

a) If an nxn matrix A is diagonalizable, then it has n L.I. eigenvectors.

b) State and prove the Spectral Theorem.

3) [15 pts] Let

$$A = \frac{1}{3}\begin{pmatrix} 1 & 8 \\ 2 & 7 \\ 2 & -2 \end{pmatrix} \qquad and \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

a) Use the Gram-Scmidt process to find an orthonormal basis for $R(A)$.

b) Factor A into QR (with the usual meaning).

c) Use b) to solve the least squares problem $Ax = b$ quickly.

Show all your work, but you can check some of your steps against some of these: $||a_1|| = 1$, $||a_2|| = 3.6$, $< a_1, a_2 >= 2$, $a_2 - 2a_1 = [2\ 1\ -2]^T$, and $a_1 - 2a_2 = [-5\ -4\ 2]^T$.

4) [20 pts] True-False. You can assume the matrices are all square.

$A^T A$ and AA^T always have the same rank.

If A and B are unitary then AB is unitary.

If A and B are singular then AB is singular.

If A is unitary then A is normal.

If A is Hermitian then A is not defective.

If A represents a rotation then A is nilpotent.

If A is Hermitian and unitary then $A^2 = I$.

If U, V and W are subspaces of R^n, and $U \perp V$ and $U \perp W$ then $V \perp W$.

If A is unitary and $\mathbf{x} \in C^n$ then $||A\mathbf{x}|| = ||\mathbf{x}||$.

If λ is an eigenvalue of a unitary matrix then $|\lambda| = 1$.

5) [20 pts] Find these matrices. One matrix might answer more than one question.

a) Let $S \subset R^3$ be span$\{\mathbf{e_1},\ \mathbf{e_1}\}$. Find the matrix representation P for the transformation $L(x) = \text{proj}\ _S(x)$.

b) Find the matrix representation M of a 120 degree counterclockwise rotation in R^2.

c) Which of your answers [to a) or b) above] are orthogonal matrices? Explain briefly.

d) Give an example of a matrix A such that $A^2 = A$ (with $A \neq O$ and $A \neq I$).

e) Give an example of a matrix A such that $A^3 = I$ (with $A \neq I$).

6) [10 pts] Let

$$\mathbf{z_1} = \frac{1}{2}\begin{pmatrix} 1+i \\ 1-i \end{pmatrix} \quad \text{and} \quad \mathbf{z_2} = \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{z} = \begin{pmatrix} 2+4i \\ -2i \end{pmatrix}$$

a) Show $\mathbf{z_1}$ and $\mathbf{z_2}$ form an orthonormal set in C^2.

b) Write the vector $\mathbf{z}$ as a linear combination of $\mathbf{z_1}$ and $\mathbf{z_2}$.