MAS 3105 Dec 13, 2006
Final Exam and Key Prof. S. Hudson

1) [10pts] Solve for X, given $XA + C = X$ and

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \qquad C = \begin{pmatrix} 0 & -1 \\ 1 & 5 \end{pmatrix}$$

2) [10pts] Use a Wronskian to show that these vectors are LI in $C[-\pi, \pi]$: e^x, e^{-x}, e^{2x}.

3) [10pts] You are a corporate spy hired to investigate student preferences in FIU math classes. After intercepting several encrypted text messages, you discover their coding matrix A, and most of their decoding matrix B. Find the missing entry of B and decode the message: 21 54 42 64 155 106 25 63 38. At the end, as usual, '1' means 'a'. Also, '0' means 'blank', etc [so, 0 thru 26 = blank, abcde fghij klmno pqrst uvwxy z]

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 2 & 3 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & -1 \\ -4 & 1 & \end{pmatrix}$$

4) [10pts] You are given three data points for (x, y): $(0, 1), (3, 4)$ and $(6, 5)$. (You may put these into a chart). Find the best least squares fit by a linear function, $y = c_0 + c_1 x$.

5) [10pts] Let $B = \{(1\ 1\ 1\ 1)^T, (1\ -1\ -1\ 1)^T, (1\ -1\ 1\ 3)^T\}$, and $S = \text{span}(B)$. Use the GS process to find an orthonormal basis of S. Note: the first two vectors are already orthogonal, and both have norm = 2.

6) [10pts] Choose ONE of these.

a) Prove thm 2.2.3, that det AB = det A det B. Use E's, and explain each step, as usual.

b) Finish the sentence and prove it (theorem 6.3.2): An nxn matrix A is diagonalizable if and only if A has . . . .

7) [15pts] Factor A into a product XDX^{-1} where D is diagonal.

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

8) [20pts] TF You can assume the matrices are all 3x3.

It is possible that $A^2 + I = O$ (the 3x3 zero matrix).
Two similar matrices must have the same rank.
If A is both unitary and Hermitian, then every eigenvalue is either -1 or 1.
If A is diagonalizable with every eigenvalue either -1 or 1, then $A^{-1} = A$.
The product of two elementary matrices is elementary.
rank BA = rank AB
If A is nonzero and diagonalizable, then it is not nilpotent.
The matrix representation of a rotation must be orthogonal.
If rank $A < n = 3$, then $\lambda = 0$ is an eigenvalue of A.
If trace $A = \lambda_1 + \lambda_2 + \lambda_3$, then A is upper triangular.

9) [5pts] Choose ONE:

a) State the parallelogram law for vectors in R^n (from MHW3, 5.1)
b) State the definition of *direct sum* of subspaces (Ch 5.2).

Bonus [5pts]: How many additions are required to compute the determinant of an nxn matrix using cofactors? Explain your formula (but you do not have to prove it carefully).

Remarks and Answers: The average was 69/100, with good results on most problems. The scores were under 65/100 on problems 2, 6 and 8 (Wronskian, Proof, TF) and very low on 9 (State/Define) which was only 5 points. The scale should be approx: A-'s start at 80, B-'s at 70, etc.

The average for the whole semester was about 75/100, but that's not firm yet, and I haven't set the scale.

1) [See Ch 1.3.12] Simple algebra leads to $X(A-I) = -C$ and $X = -C(A-I)^{-1}$, and

$$X = \begin{pmatrix} 0 & 1 \\ -1 & -4 \end{pmatrix}$$

2) [See 3.3.7d] $W(0) = -6 \neq 0$, so the functions are LI.

3) [See Ch 2.3.14] "9 0 12 15 22 5 0 12 1", or as the song goes, "I LOVE LA!" This refers, of course, to Linear Algebra.

4) [See 5.3 Ex 2] $4/3 + 2x/3$

5) [Similar to 5.6 Ex 2] The basis consists of the columns of Q;

$$Q = (1/2) \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

6) See text. For 6b), remember to explain both directions (if AND only if).

7) [See 6.3.1b] There are many answers (but the columns of your X should be multiples of mine).

$$A = XDX^{-1} = \begin{pmatrix} 6 & 6 \\ -4 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -1/2 & -1 \\ 2/3 & 1 \end{pmatrix}$$

8) FTTTF FTTTF. The first part is from 2.2.11 and was intended to be false, since the equation would imply $(\det A)^2 = -1$. But if A can have complex entries, it's true. So, I gave everybody credit for this one. The 6th one is NOT the same as the 2nd one, because one of the matrices might be singular.

9) See Ch 5. In my opinion, neither of these is a major topic, but you should know at least one of them.

Bonus) $n! - 1$. See 2.2.19. If you didn't remember this formula, hopefully you could reconstruct it from looking at 2x2, 3x3, 4x4 examples, etc.