MAS 3105 — June 22, 2006
Final Quiz and Key — Prof. S. Hudson

The unlabelled problems are 10 points each.

1) [15pts] Which of these subspaces ? (answer Yes or No to each one)

$\{(x_1, x_2, x_3)^T \mid x_1 + x_2 = 1\ \}$ (in $V = R^3$)

$\{(x_1, x_2, x_3)^T \mid x_1 = x_2 = x_3\ \}$ (in $V = R^3$)

The set of all 2x2 lower triangular matrices (in $V = R^{2x2}$).

The set of all 2x2 singular matrices (in $V = R^{2x2}$).

The set of all polynomials in P_4 of degree 2.

2) Let $S = \text{span}\ \{(1, 2, 3, 0)^T, (1, 0, 0, 1)^T\}$, a subspace of R^4. Find a basis of $S^\perp$.

3) [20pts] Use the following XDX^{-1} factorization to quickly calculate these. You do not have to simplify very much in a) to d). Numbers like 5^4 are OK. You shouldn't have to multiply matrices (but, for example, I would not accept "A^2A^5" as an answer to part b)

$$A = \begin{pmatrix} 4 & 3 & 3 \\ -5 & 4 & -5 \\ 5 & -3 & 6 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

a) A^{-1}

b) A^7

c) e^A

d) det (A)

e) Find a matrix B so that $B^2 = A$. Simplify this one completely.

4) You are given three data points for (x, y): $(0, 1), (3, 4)$ and $(6, 5)$. (You may put these into a chart). Find the best least squares fit by a linear function, $y = c_0 + c_1 x$.

5) Choose ONE of these.

a) [from MHW 6.10] Suppose you need a random symmetric 6x6 matrix in MATLAB. You set A = round(5*rand(6)), but that's not symmetric. So then you set A = ?

b) Give an example of a 3x3 defective matrix.

6) [15pts] Answer True or False: Assume A and B are 3x3 matrices.

$\det(cA) = c \det(A)$

If $\mathbf{x} \neq \mathbf{0}$ in R^3 and $A\mathbf{x} = \mathbf{0}$ then $\det(A) = 0$.

$(A - B)^2 = A^2 - 2AB + B^2$

If a list of n vectors spans R^n they must be LI.

A and A^T have the same nullity.

7) Let $B = \{(1\ 1\ 1\ 1)^T, (1\ -1\ -1\ 1)^T, (1\ 1\ 2\ 2)^T\}$, and $S = \text{span}(B)$. Use the GS process to find an orthonormal basis of S. For a little extra credit, find the QR factorization of the matrix obtained from B. Note: the first two vectors are already orthogonal, and both have norm = 2.

8) Choose ONE of these.

a) State and prove the Spectral Theorem.

b) Prove theorem 6.3.1: If $\lambda_1, \lambda_2, \ldots \lambda_k$ are distinct eigenvalues of the nxn matrix A, then the corresponding eigenvectors, $\mathbf{x_1}$, $\mathbf{x_2} \ldots \mathbf{x_k}$ are L.I. [Hint: use a proof by contradiction. Set r = rank (span(the $\mathbf{x_j}$)) and put the eigenvectors in any order you want, etc].

Answers and Remarks: The average was about 75/100, which is pretty good, and the average for the semester was about 80/100. With such high averages, I sometimes raise the scale. But after analyzing the scores on some standard questions (such as parts 2 and 4 of the final), I decided this class was clearly above average, so I actually *lowered* the scale. Well done!

1) N Y Y N N

2) Use the F.S.Thm: $S^{\perp} = R(A)^{\perp} = N(A^T)$.
A basis is $\{\ [-1,\ 1/2,\ 0,\ 1]^T,\ \ [0,\ -3/2,\ 1,\ 0]^T\ \}$

3abc) are $XD^{-1}X^{-1}$ and XD^7X^{-1} and Xe^DX^{-1}. Write these out but you don't have to multiply. 3d) is det(D) = 36. 3e) is $B = XD^{-1/2}X^{-1}$

$$B = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}. \quad \text{and} \quad e^D = \begin{pmatrix} e^4 & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & e^9 \end{pmatrix}$$

4) See 5.3 Ex2: $y = 4/3 + 2x/3$ (be sure to write the answer in equation form).

5) Several people answered both, but I only graded the first answer. 5a) I expected $A = A + A^T$ as in the MHW, but $A = A * A^T$ is also OK. There are many options for 5b. This example has $\lambda = 0$ with multiplicity 3, but just 2 LI evecs:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

6) FTFTT

7) $B = \{(1/2)(1\ 1\ 1\ 1)^T, (1/2)(1\ \ -1\ \ -1\ 1)^T, (1/2)(1\ 1\ \ -1\ \ -1)^T\}$

8) See the text. For part a) you should mention "Schur's Thm" (or "Thm 6.4.3", or at least "by a previous theorem") and should explain quickly why $A^H = A$. Mention that U stands for a unitary matrix, etc.