MAS 3105 April 26, 2007

Final Exam and Key Prof. S. Hudson

1) (10pts) Solve for X, given that $AX + B = X$ and

$$A = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} -1 & -2 \\ -3 & 0 \end{bmatrix}$$

2) (10pts) Use the Gram Schmidt process to find an orthonormal basis for R(A). Then factor $A = QR$ (orthogonal times upper triangular):

$$A = \begin{bmatrix} -1 & 3 \\ 1 & 5 \end{bmatrix}$$

3) (15pts) a) Given that A is row equivalent to U, find a basis for $N(A)$. Check !

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix} \quad U = \begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3b) Find a basis of the row space of A. Explain briefly.

3c) Find a basis of the column space of A. Explain briefly.

4) (10 pts) Show that any set of vectors that contains the zero vector must be L.D (dependent). Include the definition of LD in your proof.

5) (10pts) Let $V = C[0, \pi]$ with the usual inner product $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)\,dx$. a) Compute $||\sin x||$. b) Show that $\sin x \perp \cos x$. Show all your work.

6) (5 pts) Choose ONE:

a) Describe the eigshow utility used in several MATLAB problems in Ch6.

b) Find two LI eigenvectors in C^2 for the L which rotates vectors 90 degrees CCW in R^2.

7) (20 points) Answer True or False. Assume the matrices are 3x3, and that A is similar to B.

If A is symmetric then B is symmetric.

A and A^T have the same eigenvalues.

A and A^T have the same eigenvectors.

A and B have the same eigenvalues.

A and B have the same eigenvectors.

$A + I$ is similar to $B + I$.

det $(cA) = c$ det (A)

det $(A^k) =$ (det $(A))^k$

If A represents $L : R^3 \to R^3$ then A^2 represents $L^2 = L \circ L$.

If $L : V \to V$ is linear, $\mathbf{x} \in$ ker (L) and $v \in V$, then $L(\mathbf{x} + \mathbf{v}) = \mathbf{v}$.

8) (10pts) Let A be a 4x4 real matrix, with all 1's on the diagonal. If A is singular and $\lambda_1 = 3 + 2i$ is an eigenvalue of A, then what, if anything, can you conclude about the other three eigenvalues λ_2, λ_3 and λ_4? Explain. Hint: One must be the conjugate of λ_1.

9) (10pts) Choose ONE:

A) (6.3.1) If $\{\mathbf{x_1}, \mathbf{x_2}, \ldots \mathbf{x_k}\}$ are eigenvectors of A, from distinct eigenvalues, then they are L.I.

B) (2.2.3) Prove that det $(AB) =$ det A det B. To save a little exam time, you can assume both matrices are nonsingular.

Bonus: (5pt) Suppose A is diagonalizable. Find a simple formula for $\det(e^A)$, such as $e^{\det(A)}$ or $e^{\mathrm{tr}(A)}$, and justify it.

Remarks and Answers: The average on this final was about 64/100 (fairly normal) with very low scores only on problem 8. The grand average for the semester, including HW and MHW, was about 70/100, omitting the very low scores. The scale for the semester was adjusted about 4 points; A's = 81-100, etc.

1) This should be a very easy problem, especially since you had ones just like it on HW1, and on two previous quizzes. The average score (7.2/10) was OK, but too many people are still putting $(A - I)^{-1}$ on the wrong side. Also, there was plenty of time on this exam to CHECK your answers!

$$X = \begin{bmatrix} -5 & 2 \\ 3 & 0 \end{bmatrix}$$

2) [HW 5.6.1a] Results were OK, but a few people were quite lost, and many others did not list the basis clearly. The basis is

$$\{[-1/\sqrt{2} \quad 1/\sqrt{2}]^T, [1/\sqrt{2} \quad 1/\sqrt{2}]^T\}, \quad \text{and} \quad A = QR = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 4\sqrt{2} \end{bmatrix}$$

3) This was taken directly from a previous final exam. Brief answers:

3a) $\{[-3, -1, 1, 0, 0]^T, [-7, -3, 0, 1, 0)^T\}$ (and you should check these),

3b) List the 3 nonzero rows of U,

3c) List columns 1, 2 and 5 from A.

4) Briefly - Let $S = \{\mathbf{x_1}, \mathbf{x_2}, \ldots \mathbf{x_k}, \mathbf{0}\}$. Then $\mathbf{0} \cdot \mathbf{x_1} + \mathbf{0} \cdot \mathbf{x_2} + \ldots + \mathbf{0} \cdot \mathbf{x_k} + \mathbf{1} \cdot \mathbf{0} = \mathbf{0}$. Since the last coefficient is $1 \neq 0$, this LC is non-trivial, which implies the set is LD, by definition.

The results on this Ch 3 HW problem were not TOO bad (5.5/10) considering it was an not announced. Many answers contained decent ideas, but were very poorly organized. I did not give full credit for saying that the zero vector is a scalar multiple of any other vector - this idea could be made into a proof, but it doesn't follow the directions.

5a) $||\sin x|| = \langle \sin x, \sin x \rangle^{1/2} = 1$ (the integral is easier if you substitute $\sin^2 x = (1 - \cos 2x)/2$). 5b) Calculate that $\int_{-\pi}^{\pi} \sin x \cos x = 0$ using the substitution $u = \sin x$. I had intended to offer some integration hints during the exam and forgot - but this did not seem to affect the results much.

6a) Just describe the circle and two lines you saw, for example, to show me you did the MHW by yourself, and you were awake.
6b) Done in class (twice?). $\{[1, i]^T, [1, -i]^T\}$ (scalar multiples of these are also OK, in either order, eg $\{[i, 1]^T, [i, -1]^T\}$).

7) FTFTF TFTTT The first one is not quite obvious. But Ch 6 shows us that a random (not symmetric) 2x2 matrix B can be diagonalized. So, it is similar to a diagonal (symmetric) matrix A.

8) [p406, prob 2] The hint is that $\lambda_2 = 3 - 2i$. Since A is singular, $\lambda_3 = 0$ (mentioned in class; see also pages 302, or 309, or HW 6.1.3). The trace of A is 4, which is the sum of the e-val's (page 309). So, $\lambda_4 = -2$. Of course, the λ's can be listed in other orders.

9) See the text. For Thm 2.2.3, you should include the formula for det EA from page 102, or *at least* include the sentence 'We have already seen ...' - one of the 2 key ideas of this proof. I would also prefer that you fill in the minor gaps in the calculation on page 103 (as done in class).

Bonus: $\det e^A = \det e^D$, which is the product of the diagonal entries, $d_{ii} = e^{\lambda_1} e^{\lambda_2} \ldots e^{\lambda_n} = e^{\text{tr} A}$.