MAS 3105 — Dec 10, 2008
Final Exam and key — Prof. S. Hudson

1) (10pts) Suppose A is nxn, real, and skew-symmetric (so, $A^T = -A$). Show that:
a) A is normal.

b) If n is odd, A must be singular.

2) (10 pts) a) Find the least squares solution to $A\mathbf{x} = \mathbf{b}$, using this QR factorization:

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & -4 & 2 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{where} \quad \mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

2b) Let $S = R(A)$ and use part a) to find $\text{proj}_S \mathbf{b}$.

3) (10pts) Choose ONE (a fairly short proof)

a) State and prove the Spectral Theorem
b) If $Y \subset V$ is a subspace, then so is $Y^\perp$.
c) If A and B are nonsingular, then so is AB (state and prove a formula for its inverse).

4) (10pts) Factor A into XDX^{-1} where D is diagonal (compute the 3 factors - and I'd suggest checking them by multiplication).

$$A = \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix}$$

5) (10pts) Let $S \subset R^{3x3}$ be the vector space of all symmetric 3x3 matrices. Find dim(S) and explain briefly.

6) (5pts) Suppose A is 2x2 and tr(A)= 3 and det(A) = 2. Find the 2 real eigenvalues of A.

7) (10 pts) Let $S = \{1, x\}$ be a basis of P_2. Find the matrix representation of $L : P_2 \to P_2$, wrt S, where $L(p) = p'(x) + x \cdot p(0)$.

8) (20 points) Answer True or False. Assume all matrices have the right shapes to be added or multiplied.

Every 3x3 matrix A has at least one real eigenvalue.

Every 4x4 matrix A has at least one real eigenvalue.

$\langle p, q\rangle = p(0)q(0) + p(1)q(1)$ defines an inner product on P_3.

$f(\mathbf{x}) = ||\mathbf{x}||_1 + ||\mathbf{x}||_2$ defines a norm on R^2.

A defective nxn matrix cannot have have n distinct eigenvalues.

The product of two Hermitian matrices is Hermitian.

The sum of two Hermitian matrices is Hermitian.

The sum of two nonsingular matrices is nonsingular.

If A is diagonalizable and B is not, then A cannot be similar to B.

If 0 is an eigenvalue of A, then the nullity of A is positive.

9) (15pts) Choose ONE:

A) Prove that det AB = det A det B .

B) Prove that if A and B are nxm and mxn (resp) and $AB = I$ and $BA = I$, then $n = m$.

C) An nxn matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Bonus (5pts): Find an orthonormal basis of $N(A)$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Remarks and Answers: The results were pretty good on most problems, except 1 and 5 and maybe 9. The average was about 60-65 of 100.

1a) $A^H A = A^T A = -A^2$ and $AA^H = AA^T = -A^2$ so $A^H A = AA^H$ and A is normal.

1b) $\det(A) = \det\,(-A^T) = \det(-A) = (-1)^n \det(A) = -\det(A)$, since n is odd. Add det to both sides and get 2 $\det(A) = 0$, and A is singular. (this is an exercise from Ch 2).

2a) As usual, reduce the normal eqns to $R\hat{x} = Q^T b = [-1 \quad -1 \quad 2]^T$. From back sub, get $\hat{x} = [-2/5 \quad 0 \quad 1]^T$.

2b) $A\hat{x} = [3/5 \quad 1/5 \quad 6/5 \quad -8/5]^T$

3) These are all in the text. Part c) should be very easy (check that $(AB)^{-1} = B^{-1}A^{-1}$). But most people chose part a), since it was advertised, and they did OK.

4) Compute the e-things and get

$$D = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \quad X = \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix}$$

Then compute X^{-1} (probably using the adjoint) and check.

5) This was supposed to be easy. Just write down a typical element of S and count the letters (the reasoning was explained in class) to get dim = 6:

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}$$

6) $\lambda = 1, 2$

7) Compute $L(1) = x$ and $L(x) = 1$ to get

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

8) TFFFT FTFTT . The first one is true because every polynomial of odd degree has a real root. The 3rd fails because $\langle p, p \rangle = 0$ when $p(x) = x(x-1)$ (example from class). The 4th fails the $\alpha\mathbf{x}$ test. The others have been explained in class or on other exams.

9) Parts a and c are in the book. You can prove part b) fairly easily using ranks (but I don't know any other easy way to do it).

10) Apply GSO to the 2 vectors you get from GE ($[1 \ -1 \ 0]^T$ and $[1 \ 0 \ -1]^T$ for example).