

1) (10pts) Find e^A Compute each entry; you can leave those in unsimplified forms, such as $e + \pi + \frac{3}{6}$:

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 1 \\ 15 & 4 & 4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & -1 & 0 \\ -15 & 4 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

2) (10pts) For the matrix A in problem 1), explain why $A^2 = A$ must be true, without actually multiplying out $A * A$. Then, find a matrix $B \neq A$ so that $B^2 = A$.

3) (15 pts) a) Find the least squares solution to $A\mathbf{x} = \mathbf{b}$, using this QR factorization:

$$A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & -4 & 2 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{where } \mathbf{b} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$$

3b) Let $S = R(A)$. Find $\text{proj}_S \mathbf{b}$.

4) (10pts) Find the standard matrix representation for the following linear transformation: L reflects each vector about the line $x_1 = x_2$, and then projects it onto the x_1 axis.

5) (10pts) Show that the product of two unitary matrices is unitary. [Hint - use the formula with U^H .]

6) [15pts] Factor A into a product DXD^{-1} where D is diagonal.

$$A = \begin{pmatrix} 5 & 6 \\ -2 & -2 \end{pmatrix}$$

7) (20 points) Answer True or False. Assume the matrices are 3x3 (if necessary).

For every A , the matrices AA^T and $A^T A$ are both symmetric.

If M is a 5x3 matrix, then $Mx = 0$ has nontrivial solutions.

An inconsistent system $A\mathbf{x} = \mathbf{b}$ has a unique Least Squares solution.

Every nonsingular matrix is diagonalizable.

If V has a spanning set of 5 vectors then $\dim(V) \leq 5$.

The product of two Hermitian matrices is Hermitian.

The number of nonzero eigenvalues is the rank of A .

Every unitary matrix is normal.

If A and B have the same eigenvalues 2, 4 and 8, then A is similar to B .

If 0 is an eigenvalue of the 3x3 matrix A , then $\text{rank}(A) \leq 2$.

8) (10pts) Choose ONE of these textbook proofs:

A) Prove that any two similar square matrices A and B must have the same eigenvalues.

B) State and prove the Spectral Theorem (6.4.4).

Bonus (5 pts): Give examples of two similar 3x3 matrices A and B such that $\text{rank } A \neq \text{rank } B$ (or prove this is not possible).

Remarks and Answers: Several PM people took the AM exam. Also, I passed out a few PM exams to the AM class. If this exam looks unfamiliar, check my PM final exam key instead. As far as I know, the two exams were about equal in difficulty. The AM average was about 65/100, but I haven't finished analyzing this, and haven't set a scale.

Thank you for your help in finding some missing grades, which were eaten by my computer. I think everything is in order now, except for one or two students who didn't save their quizzes. This problem shouldn't happen often, but you should always save all your work. Please email me ASAP if you have any more info that you did not give me in finals week.

1) $e^A = X e^D X^{-1}$ and e^D is easy (the diagonal entries are e , e and 1).

$$e^A = \begin{bmatrix} e & 0 & 1-e \\ 0 & e & 4-4e \\ 0 & 0 & 1 \end{bmatrix}$$

2a) $A^2 = XDX^{-1} \cdot XDX^{-1} = XD^2X^{-1} = XDX^{-1} = A$. We get D^2 by squaring the diagonal entries, but that has no effect in this example, since $1^2 = 1$ and $0^2 = 0$.

2b) The simplest answer is to set $B = -A$ ($B^2 = (-A)^2 = A^2 = A$). But that won't always work. I expected something like $B = XCX^{-1}$ where $C \neq D$ and $C^2 = D$. For example, let

$$C = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and then compute B from that.

3a) Set $R\hat{\mathbf{x}} = Q^T \mathbf{b}$ (simplified normal equations). Include the $1/5$ into Q and Q^T ; otherwise it is not orthogonal. Then use back substitution. Many people computed R^{-1} instead, but that's messier. Likewise using $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$ is OK, but messier. Get $\hat{\mathbf{x}} = [-1, -1, 2]^T$.

3b) $\mathbf{p} = A\hat{\mathbf{x}} = [-7/5, 4/5, 6/5, -8/5]^T$. If you have time (and most people left a few minutes early), check your work. For example, it's fairly easy to check that $\mathbf{b} - \mathbf{p} \perp \mathbf{p}$.

4) It's probably simplest to get $L(e_1) = [0 \ 0]^T$ and $L(e_2) = [1 \ 0]^T$ from a picture (or even a mental picture, but remember show some work). Then combine these columns to get the matrix A . Another approach is to view L as a composite, which leads to the same answer:

$$A = A_2 A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

5) Assume U and V are unitary. So, $U^H U = I$ and $V^H V = I$, based on the remarks following the definition (I insisted on some comment here, for full credit. I accepted 'by definition' or 'by a theorem'). Then $(UV)^H (UV) = V^H U^H UV = V^H IV = I$ which shows UV is unitary.

6) The eigenvalues are 2 and 1, which go into D . The eigenvectors are $[2, -1]^T$ and $[-3, 2]^T$ which go into X . There are other possible answers. Again, it's a good idea to check yours.

$$A = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

7) TFFFT FFTTT. Statement 4 is false because nonsingular matrices can be defective; see Example 3, page 326. Statement 7 is also false because of defective examples, such as the nilpotent one below (both examples were covered in class). A has no nonzero eigenvalues, but has rank = 2.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

8) See the textbook. The results were OK, but some of the proofs seemed badly memorized, without proper understanding.

Bonus: This is not possible, based on HW 3.6.16, which implies $\text{rank}(A) = \text{rank}(S^{-1}BS) = \text{rank}(B)$. Some people tried to prove this one using TF statement 7. Nice idea, but that statement is false! Another faulty plan is to assume A and B are diagonalizable (which might be false). Then they'd have the same λ 's and the same D , but it is still not quite clear that they have the same rank.