MAS 3105 — PM: April 22, 2008
Final Exam and Key — Prof. S. Hudson

1) [20 pts] a) Confirm [showing all work] that the columns of V are mutually orthogonal in C^2, but are not unit vectors.

$$V = \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$$

b) Apply Gram-Schmidt (a very quick and easy version) to find a unitary matrix U and a diagonal matrix R so that $V = UR$.

c) Let $B = \{\mathbf{v_1}, \mathbf{v_2}\}$ be the basis of C^2 we get from the columns of V. Find the transition matrix T from the standard basis S to B.

d) Is V a normal matrix ? Prove your answer.

2) [10 pts] Let $S = \text{span}\ \{(1,2,3,0)^T, (1,0,0,1)^T\}$, a subspace of R^4. Find a basis of $S^\perp$.

3) [10 pts] Find two LI eigenvectors for A.

$$A = \begin{bmatrix} 2 & -8 \\ 1 & -4 \end{bmatrix}$$

4) [10 pts] Let A be a diagonalizable matrix whose eigenvalues are all either 1 or -1. Show that $A = A^{-1}$.

5) [10 pts] Find the matrix representation of $L : R^2 \to R^2$, where L is projection onto the line $2x_1 = x_2$.

6) [20 points] Answer True or False:

If 0 is an eigenvalue of A, then A is singular.
Every real unitary matrix is orthogonal.
Every diagonal matrix is normal.
Every real symmetric matrix is diagonalizable.
Every real 3x3 matrix has at least one real eigenvalue.
If A and B are similar nxn matrices, then det A = det B.
Every square matrix A can be factored, $A = QR$ into an orthogonal matrix times an upper triangular matrix.
If A and B have the same eigenvalues 2, 4 and 8, then A is similar to B.
If A is a 7x5 matrix, then $Ax = 0$ has nontrivial solutions.
If A is nxn and singular, then its nullity is positive.

7) [10pts] In $V = C[-\pi, \pi]$, let S be the subspace spanned by $L = \{1,\ \cos(2x),\ \cos^2(x)\}$.

a) Determine whether L is LI using a Wronskian. Hints: $\sin(2x) = 2\sin(x)\cos(x)$ and $\sin^2(x) = (1 - \cos(2x))/2$.

7b) Find the dimension of S, and explain briefly.

8) [10 pts] Choose ONE:

A) Prove that $N(A) = R(A^T)^\perp$.
B) State and prove the Spectral Theorem (6.4.4).
C) An nxn matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Small Bonus (about 5pts): Give an example of a 3x3 defective matrix. Justify briefly.

Remarks and Answers: This was a 2 hour exam, per the new FIU exam schedule. This exam includes several problems from my Summer 2002 final. The scores were rather low on problems 1, 4 and especially 5 (which was pretty simple, except that it required skills from both Ch 4 and Ch 5). The average was about 60 to 65.

1a) Calculate $v_i^H v_i = 2 \neq 1$, and $v_1^H v_2 = 0$. For example

$$v_1^H v_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix} [1 \quad -i] = 1 - 1 = 0$$

1b) From 1a), both norms are $\sqrt{2}$ which helps us normalize the columns of V, and also gives us $r_{11} = r_{22}$. Since they are already orthogonal, the projection is zero, so

$$V = UR = \sqrt{1/2} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \cdot \sqrt{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1c) $T = V^{-1} = R^{-1}U^{-1}$. Since U is unitary, $U^{-1} = U^H$ which is easy. With 1b) we get (check!)

$$V^{-1} = (1/2) \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}$$

1d) Yes. It is a constant times a unitary matrix, so $V^H V = c^2 U^H U = c^2 I$, and VV^H is the same.

2) Use the Fund.SSP.Thm; $N(A^T)$ is spanned by $[-2\ 1\ 0\ 2]^T$ and $[-3\ 0\ 1\ 3]^T$ and these are LI.

3) $p(\lambda) = \lambda(\lambda + 2)$ so $\lambda_1 = 0$ and $\lambda_2 = 2$. Get $\mathbf{x_1} = [4\ 1]^T$ and $\mathbf{x_2} = [2\ 1]^T$ (for example).

4) We are given $A = XDX^{-1}$, where the diagonal entries of D are all either 1 or -1. The reciprocals of these numbers are themselves, so D is nonsingular and $D^{-1} = D$. So, $A^{-1} = XD^{-1}X^{-1} = XDX^{-1} = A$.

5) It is possible to use the normal equations for this, but I expected you to use the standard Ch 4 method. The line is spanned by $v = [1\ 2]^T$. The first column of A is

$$L(e_1) = \text{proj}_v(e_1) = \frac{e_1^T v}{||v||^2} v = (1/5)v$$

Likewise, the second column is $(2/5)v$, so $A = [1/5\ 2/5; 2/5\ 4/5]$.

6) TTTTT TFTFT. Part 6 was HW (take det of both sides of $A = SBS^{-1}$). Part 7 is true if rank $(A) = n$. Part 9 would be true for a 5x7 matrix (under-determined, implying free variables).

7) $W = 0$ so they are LD. This implies $dim(S) < 3$, but the first two vectors in L are clearly LI, so $dim(S) = 2$.

8) See the text (I think B is the easiest).

Bonus: A nilpotent matrix must be defective (this idea avoids any lengthy justification). For example,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$